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GROUND-WATER HYDROLOGY 561

THEORY OF GROUND-WATER MOTION

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NMIMT

Theory of Ground Water Motion GWH 561

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Basic Flow Formula; Darcy's Law

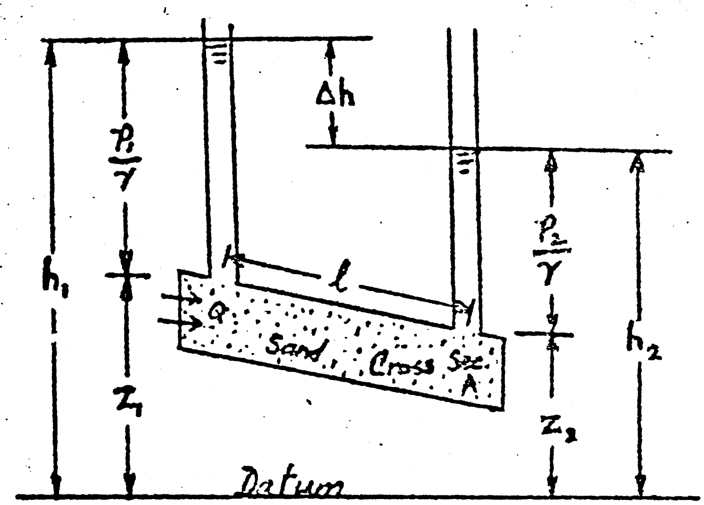
The energy or as it is more commonly called the head in a flow system is given by Bernoulli's equation.

$$h = \frac{v^2}{2g} + \frac{p}{\gamma} + z$$

In groundwater flow systems the velocities are so small that the velocity head can be neglected and the equation reduces to:

$$h = \frac{p}{\gamma} + z$$

Darcy by running an experiment similar to the one illustrated established the Darcy's Law:



$$Q = KA\frac{\Delta h}{l}$$

or

$$v_{av} = \frac{Q}{A} = K\frac{\Delta h}{l}$$

where K is the "hydraulic conductivity" depending on properties of both the solid and fluid material

$$K = c\frac{d^2\gamma}{\mu}$$

where cd^2 = coefficient of permeability

d = effective grain size (D_{10})(10% passing)

The Generalized Darcy's Law can be written as

$$v_s = -K_s \frac{\partial h}{\partial s}$$

$$= -\frac{\partial \phi}{\partial s}$$

where

$\phi = Kh =$ Velocity potential. (s) denotes any general direction.

The (-) sign is conventional, indicating positive velocities in the positive s-direction.

In Cartesian Coordinates:

$$v_x = -K_x \frac{\partial h}{\partial x}$$

$$v_y = -K_y \frac{\partial h}{\partial y}$$

$$v_z = -K_z \frac{\partial h}{\partial z}$$

$$v_s = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

In Polar Coordinates:

$$v_r = -K_r \frac{\partial h}{\partial r}$$

$$v_\theta = -K_\theta \frac{1}{r} \frac{\partial h}{\partial \theta}$$

Darcy's Law is applicable only to laminar flow.

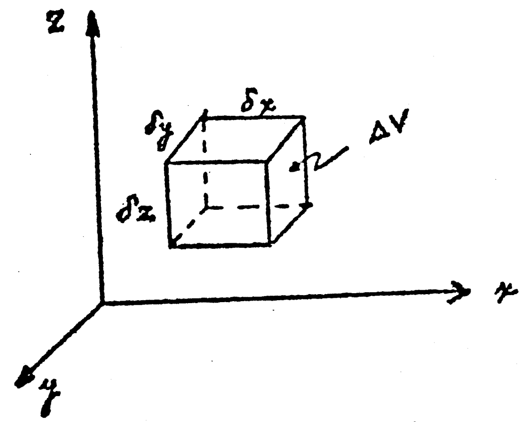
Reynold's Number $= \frac{vd}{\nu} \leq 1$ laminar

≥ 10 turbulent

$1 < R < 10$ unstable or transitional

In porous media $R < 1$ and Darcy's Law applies.

Derivation of General Diff. Equation of Ground-Water Flow



Inflow = Outflow + ΔStorage

the hydrologic equation, which is merely a statement of the law of conservation of matter.

It can also be written as

Inflow − Outflow = ΔStorage.

Consider an elemental volume, ΔV, in a porous media, where there is a flow.

Inflow:

In terms of mass per unit time:

$$\text{Inflow} = Q \times \rho = \frac{V}{t} \times \frac{m}{V} = \frac{m}{t} = Av\rho$$

Inflow in x − direction: $\rho v_x \delta z \delta y$

" " y − " : $\rho v_y \delta x \delta z$

" " z − " : $\rho v_z \delta x \delta y$

Total Inflow: $\rho v_x \delta y \delta z + \rho v_y \delta x \delta z + \rho v_z \delta x \delta y$

Outflow: The velocity and sometimes the density changes from one side to the other therefore

Outflow in x − direction: $\left(\rho v_x + \frac{\partial(\rho v_x)}{\partial x}\,\delta x\right)\delta y \delta z$

" " y − " : v_y +

" " y − " : $\left(\rho v_y + \frac{\partial(\rho v_y)}{\partial y}\,\delta y\right)\delta x \delta z$

Outflow in the z - direction: $\left(\rho v_z + \frac{\partial(\rho v_z)}{\partial z}\,\delta z\right)\delta x \delta y$

Total Outflow:

$$\left(\rho v_x + \frac{\partial(\rho v_x)}{\partial x}\,\delta x\right)\delta y \delta z + \left(\rho v_y + \frac{\partial(\rho v_y)}{\partial y}\,\delta y\right)\delta x \delta z + \left(\rho v_z + \frac{\partial(\rho v_z)}{\partial z}\,\delta z\right)\delta x \delta y$$

Δ Storage:

The change in storage is equal to the change in the mass of water present in the element, with respect to time:

$$\Delta \text{ Stor.} = \frac{\partial(\delta M)}{\partial t}$$

$$\delta M = \rho\theta\delta x\delta y\delta z$$

where θ = porosity.

Since the greatest change in the dimensions of the element, due to compression or expansion, will occur in the z - direction, we may consider δx and δy as constants.

$$\frac{\partial(\delta M)}{\partial t} = \frac{\partial(\rho\theta\delta x\delta y\delta z)}{\partial t} = \left(\theta\delta z\frac{\partial\rho}{\partial t} + \rho\delta z\frac{\partial\theta}{\partial t} + \rho\theta\frac{\partial(\delta z)}{\partial t}\right)\delta x\delta y$$

Bulk Modulus of sand $= \frac{1}{\alpha} \doteq -\frac{d\sigma_z}{\left(\frac{d(\delta z)}{\delta z}\right)}$ { negative because size decreases with increasing σ_z }

α = Compressibility (vertical) of sand $= \frac{1}{E}$

$d\sigma_z$ = Change in stress

$\frac{d(\delta z)}{\delta z}$ = Change in strain

$$d(\delta z) = -\alpha \delta z d\sigma_z$$

$$\frac{\partial(\delta z)}{\partial t} = -\alpha \delta z \frac{\partial(\sigma z)}{\partial t}$$

V_s = Volume of solids = $(1 - \theta)\ \delta x \delta y \delta z$ = constant

$$dV_s = d[(1-\theta)\delta x \delta y \delta z] = 0$$

$$\delta z\ d(1-\theta) + (1-\theta)\ d(\delta z) = 0$$

$$\frac{\partial \theta}{\partial t} = \frac{(1-\theta)}{\delta z} \frac{\partial(\delta z)}{\partial t}$$

but $\frac{\partial(\delta z)}{\partial t} = -\alpha \delta z \frac{\partial(\sigma_z)}{\partial t}$

Therefore $\frac{\partial \theta}{\partial t} \equiv -(1-\theta)\,\alpha \frac{\partial(\sigma_z)}{\partial t}$

Bulk Modulus of water $= \frac{1}{\beta} = \frac{dp}{d\rho/\rho_o}$

β = Compressibility of fluid $= \frac{1}{E}$

dp = Change in pressure

$d\rho$ = Change in density

ρ_o = Original density

$$d\rho = \rho_o \beta dp$$

$$\frac{\partial \rho}{\partial t} = \rho_o \beta \frac{\partial p}{\partial t}$$

Since the element is in static equilibrium.

$$p + \sigma_z = \text{Vertical load [D.L. + Atm. Pr.]}$$

$$= \text{Constant}$$

$$dp = d\sigma_z$$

$$\frac{\partial p}{\partial t} = -\frac{\partial \sigma z}{\partial t}$$

Therefore

$$\frac{\partial(\delta z)}{\partial t} = \alpha \delta z \frac{\partial p}{\partial t} \quad ; \quad \frac{\partial \theta}{\partial t} = (1-\theta)\alpha \frac{\partial p}{\partial t} \quad ; \quad \frac{\partial \rho}{\partial t} = \rho_o \beta \frac{\partial p}{\partial t}$$

$$\frac{\partial(\delta M)}{\partial t} = \left(\theta \delta z \rho_o \beta \frac{\partial p}{\partial t} + \rho \delta z (1-\theta)\alpha \frac{\partial p}{\partial t} + \rho \theta \alpha \delta z \frac{\partial p}{\partial t}\right) \delta x \delta y$$

$$= (\theta \rho_o \beta + \rho\alpha - \rho\alpha\theta + \rho\alpha\theta) \frac{\partial p}{\partial t} \; \delta x \delta y \delta z$$

$$= (\theta \rho_o \beta + \rho\alpha) \frac{\partial p}{\partial t} \; \delta x \delta y \delta z$$

Putting the results in the equation:

Inflow − Outflow = Δ Storage

we obtain

$$-\left(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right)\delta x\delta y\delta z = (\theta\rho_o\beta + \rho\alpha)\,\frac{\partial p}{\partial t}\,\delta x\delta y\delta z$$

$$-\left(\rho\,\frac{\partial v_x}{\partial x} + v_x\frac{\partial\rho}{\partial x} + \rho\frac{\partial v_y}{\partial y} + v_y\frac{\partial\rho}{\partial y} + \rho\,\frac{\partial v_z}{\partial z} + v_z\frac{\partial\rho}{\partial z}\right) = (\theta\rho_o\beta + \rho\alpha)\,\frac{\partial p}{\partial t}$$

But

$$v_s = -K_s\frac{\partial h}{\partial s} \qquad \text{and} \qquad \frac{\partial v_s}{\partial s} = -K_s\frac{\partial^2 h}{\partial s^2}$$

and $\quad p = \gamma h \quad \therefore \quad \frac{\partial p}{\partial t} = \gamma\frac{\partial h}{\partial t}$

$$\left(\rho K_x\frac{\partial^2 h}{\partial x^2} + K_x\frac{\partial h}{\partial x}\frac{\partial\rho}{\partial x} + \rho K_y\frac{\partial^2 h}{\partial y^2} + K_y\frac{\partial h}{\partial y}\frac{\partial\rho}{\partial y} + \rho K_z\frac{\partial^2 h}{\partial z^2} + K_z\frac{\partial h}{\partial z}\frac{\partial\rho}{\partial z}\right) = (\theta\rho_o\beta + \rho\alpha)\gamma\,\frac{\partial h}{\partial t}$$

The change in the density is very small and we can assume $\partial\partial\rho = 0$, i.e $\rho_o = \rho =$ constant. Also we assume that $K_x = K_y = K_z = K$ isotropic

$\therefore$

$$\rho K\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = \rho\gamma(\theta\beta + \alpha)\,\frac{\partial h}{\partial t}$$

$$\nabla^2 h = \frac{\theta\beta\gamma}{K}\left(1 + \frac{\alpha}{\theta\beta}\right)\frac{\partial h}{\partial t}$$

$$\nabla^2 h = \frac{S_s}{K}\frac{\partial h}{\partial t} \qquad \text{Gen. Eq. for Gr. W. flow}$$

where

$S_s = \theta\beta\gamma\left(1 + \frac{\alpha}{\theta\beta}\right)$ = Specific storage = The amount of water which a unit volume of the aquifer releases from or gains to storage under a unit decline or rise of head.

$\theta\beta\gamma$ is storage due to compressibility of water

$\alpha\gamma$ " " " " " " aquifer

If ρ is not constant

$$\partial\rho = \rho_0\beta\partial p = \rho_0\beta\gamma\partial h$$

and

$$K_x \frac{\partial h}{\partial x}\frac{\partial \rho}{\partial x} = K_x\rho_0\gamma\beta\left(\frac{\partial h}{\partial x}\right)^2$$

The gradient is usually small and $\left(\frac{\partial h}{\partial x}\right)^2$ is very small. Furthermore it is multiplied by β , which is also very small for water, and therefore the whole term can be neglected. The resulting equation will be the same as before.

<u>Changes in the Gen. Diff. Eq. due to changes in barometric pressure, tidal fluctuations, etc.</u>

In calculating $\frac{\partial(\delta M)}{\partial t}$ we assummed that

$$p + \sigma_z = \text{Constant dead load}$$

If one of the parameters of the total [redacted] load, for example, atmospheric pressure is variable, then

$$p + \sigma_z = \text{Const.} + p_a$$

$$\sigma_z = p_a + \text{Const} - p$$

$$\frac{\partial\sigma_z}{\partial t} = \frac{\partial p_a}{\partial t} - \frac{\partial p}{\partial t}$$

By taking into consideration the effect of the change in the p_a or in any of the other variables forming the total load, equations describing the flow can be derived.

However, as the resulting equations are difficult to solve it is preferable to first solve the already derived Gen. Diff. Eq. for constant total load and then apply a correction to compensate for the variable load term. This correction will be the effect of the variable load to h, i.e. in the case of variable atmospheric pressure, $\frac{dh}{dp_a}$ will be the correction to be applied to h obtained by solving the Gen. Diff. Eq..

To illustrate, corrections due to atmospheric pressure changes, and tidal fluctuations, will be derived below.

<u>Fluctuation of water levels in response to atmospheric pressure changes.</u>

$$p + \sigma_z = \text{D.L. } p_a$$

$$dp + d\sigma_z = dp_a$$

But
$$p = p_a + \gamma h$$

$$dp = dp_a + \gamma dh$$

$$\gamma \frac{dh}{dp_a} = \frac{dp-dp_a}{dp_a}$$

or

$$\gamma \frac{dh}{dp_a} = \frac{dp-dp-d\sigma_z}{dp+d\sigma_z} = -\frac{d\sigma_z}{dp+d\sigma_z} = -\frac{d\sigma_z/dp}{1+\frac{d\sigma_z}{dp}}$$

Consider a volume of the aquifer, V.

$$V = V_{water} + V_{solids} \text{ (constant)}$$

$$dV = dV_w$$

and $$V_w = \theta V$$

$$\therefore \quad \frac{dV}{\theta V} = \frac{dV_w}{V_w}$$

Bulk Modulus of water $= \frac{1}{\beta} = -\frac{dp}{\left(\frac{dV_w}{V_w}\right)}$

Bulk Modulus of sand $= \frac{1}{\alpha} = -\frac{d\sigma_z}{\left(\frac{dV}{V}\right)} = -\frac{d\sigma_z}{\theta\left(\frac{dV_w}{V_w}\right)}$

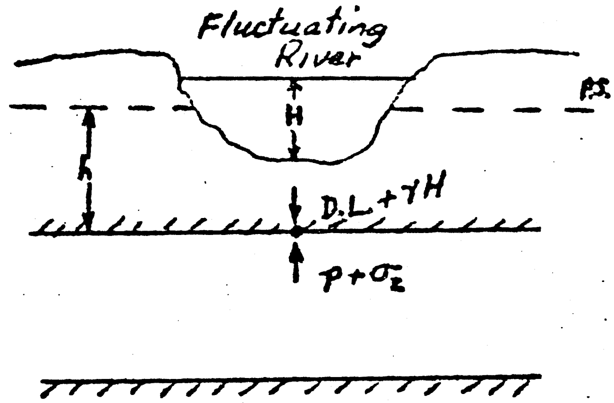
$$\frac{d\sigma_z}{dp} = \frac{\theta\beta}{\alpha}$$

$$\gamma \frac{dh}{dp_a} = -\frac{\theta\beta/\alpha}{1+\frac{\theta\beta}{\alpha}}$$

$$\frac{dh}{dp_a} = -\frac{1}{\gamma\left(1+\frac{\alpha}{\theta\beta}\right)}$$

Fluctuation of water levels in response to tidal fluctuations'

$p + \sigma_z = \text{Const. load} + \gamma H$



$dp + d\sigma_z = \gamma dH$

$$\frac{1}{\gamma}\frac{dp}{dH} = \frac{dp}{dp+d\sigma_z}$$

But $\quad h = \frac{p}{\gamma} + z \text{ (const.)}$

$$dh = \frac{1}{\gamma}\, dp$$

$$\frac{dh}{dH} = \frac{dp/d\sigma_z}{1 + \frac{dp}{d\sigma_z}}$$

$$\frac{dp}{d\sigma_z} = \frac{\alpha}{\theta\beta}$$ (From the previous derivation)

$$\frac{dh}{dH} = \frac{\alpha}{\alpha+\theta\beta}$$

Diff. Eq. in Anisotropic Media

If the hydraulic conductivities in the x, y, and z direction are different the Gen. Diff. Eq. will have the form

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t}$$

which is more complicated to solve.

By substituting

$$x = \sqrt{K_x}\ x' \ ; \ y = \sqrt{K_y}\ y' \ ; \ z = \sqrt{K_z}\ z'$$

we obtain

$$\frac{\partial\big(f(x')\big)}{\partial x} = \frac{df(x')}{dx'} \cdot \frac{dx'}{dx} = \frac{df(x')}{dx'} \cdot \frac{1}{\sqrt{K_x}}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f(x')}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{df(x')}{dx'} \cdot \frac{1}{\sqrt{K_x}}\right] = \frac{d}{dx'}\left[\frac{1}{\sqrt{K_x}}\ \frac{df(x')}{dx'}\right]\frac{dx'}{dx} = \frac{1}{K_x}\ \frac{\partial^2 f(x')}{\partial x'}$$

Therefore

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{K_x}\ \frac{\partial^2 h}{\partial x'^2}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{1}{K_y}\ \frac{\partial^2 h}{\partial y'^2}$$

$$\frac{\partial^2 h}{\partial z^2} = \frac{1}{K_z}\ \frac{\partial^2 h}{\partial z'^2}$$

Substituting in the Diff. Eq.

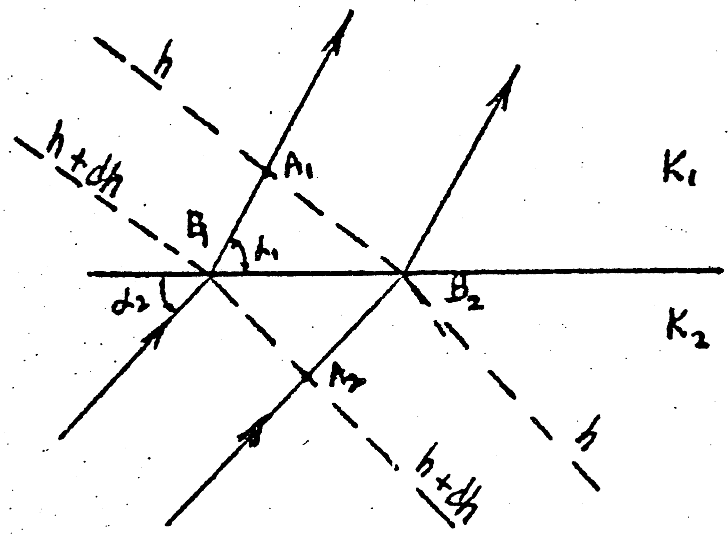
$$\frac{\partial^2 h}{\partial x'^2} + \frac{\partial^2 h}{\partial y'^2} + \frac{\partial^2 h}{\partial z'^2} = S_s\ \frac{\partial h}{\partial t}$$

Stream lines crossing a bed of different hydraulic conductivity

The discharge

$$Q = KA \frac{dh}{d\ell}$$

is constant between the two stream lines



$$K_2 \cdot \overline{A_2B_1} \frac{dh}{A_2B_2} = K_1 \cdot \overline{A_1B_2} \frac{dh}{A_1B_1}$$

But $\dfrac{\overline{A_2B_1}}{A_2B_2} = \tan \alpha_2$

and $\dfrac{\overline{A_1B_2}}{A_1B_1} = \tan \alpha_1$

Therefore $$\frac{K_2}{K_1} = \frac{\tan \alpha_1}{\tan \alpha_2}$$

If $K_2 << K_1$ then $\alpha_1 = 0°$

Boundary Conditions

To solve the Gen. Diff. Eq. a certain number of boundary conditions are necessary. Any physical condition that has an effect on the flow system can be used as a boundary condition. For example:

1. An impermeable layer, $K \frac{\partial h}{\partial n} = 0$
2. A constant head at a known point

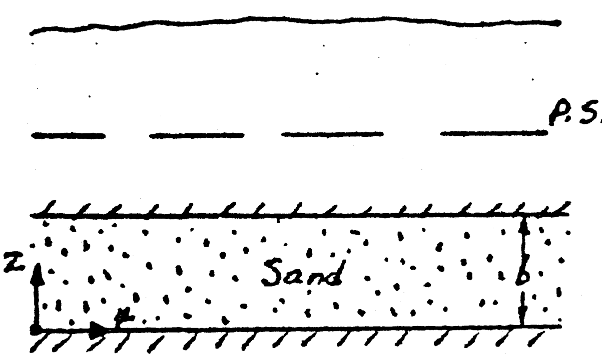
3. $h = f(t)$ at a known point

4. A constant discharge or drawdown

and many others which can be encountered in different cases.

Equations for Special Cases

Equation for flow in horizontal confined aquifers of uniform thickness



The average head along a column of the aquifer is almost equal to the height of the piezometric surface. Therefore if we are interested in the shape of the piezometric surface, rather than the head at individual points of the aquifer we can obtain a simpler equation for this purpose.

Since our objective is to average the head along a column of the aquifer, we can sum the heads at the individual points of the column by integrating the general flow equation with respect to z, from 0 to b, and then average it by dividing this sum by b.

$$\int_0^b \frac{\partial^2 h}{\partial x^2}\,dz + \int_0^b \frac{\partial^2 h}{\partial y^2}\,dz + \int_0^b \frac{\partial^2 h}{\partial z^2}\,dz = \frac{S_s}{K}\int_0^b \frac{\partial h}{\partial t}\,dz$$

But

$$\frac{\partial}{\partial x}\int_{f_1(x)}^{f_2(x)} F(\alpha,y)\,d\alpha = F\big(f_2(x),y\big)\frac{\partial}{\partial x}f_2(x) - F\big(f_1(x),y\big)\frac{\partial}{\partial x}f_1(x) + \int_{f_1(x)}^{f_2(x)}\frac{\partial}{\partial x}F(\alpha,y)\,d\alpha$$

Since $f_1(x)$ and $f_2(x)$ are constants, 0 and b,

$$\frac{\partial}{\partial x}\int_0^b F(\alpha,y)d\alpha = \int_0^b \frac{\partial}{\partial x} F(\alpha,y)d\alpha$$

and

$$\frac{\partial^2}{\partial x^2}\int_0^b F(\alpha,y)d\alpha = \int_0^b \frac{\partial^2}{\partial x^2} F(\alpha,y)d\alpha$$

Therefore the equation becomes

$$\frac{\partial^2}{\partial x^2}\int_0^b h\,dz + \frac{\partial^2}{\partial y^2}\int_0^b h\,dz + \int_0^b \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right)dz = \frac{S_s}{K}\frac{\partial}{\partial t}\int_0^b h\,dz$$

If we define the average head $\bar{h} = \frac{1}{b}\int_0^b h\,dz$ then $\int_0^b h\,dz = b\bar{h}$

and

$$\frac{\partial^2}{\partial x^2}(b\bar{h}) + \frac{\partial^2}{\partial y^2}(b\bar{h}) + \left.\frac{\partial h}{\partial z}\right|_b - \left.\frac{\partial h}{\partial z}\right|_0 = \frac{S_s}{K}\frac{\partial}{\partial t}(b\bar{h})$$

Noting that $\frac{\partial h}{\partial z} = -\frac{v_z}{K} = 0$ both at 0 and b, and dividing by b, we finally have:

$$\frac{\partial^2\bar{h}}{\partial x^2} + \frac{\partial^2\bar{h}}{\partial y^2} = \frac{S_s}{K}\frac{\partial\bar{h}}{\partial t}$$

or

$$\frac{\partial^2\bar{h}}{\partial x^2} + \frac{\partial^2\bar{h}}{\partial y^2} = \frac{S}{T}\frac{\partial\bar{h}}{\partial t}$$

where

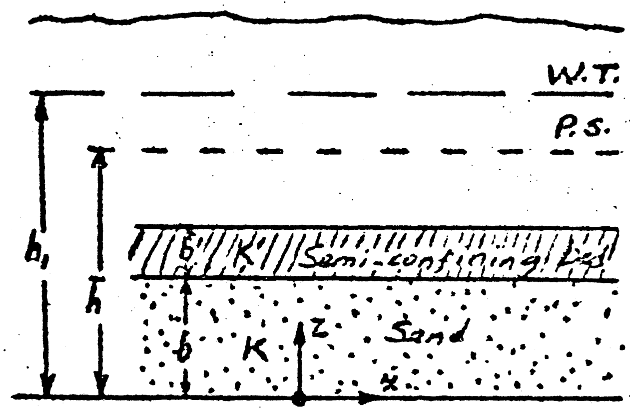
$S = S_s b$ = Coefficient of storage = Volume of water that a column of the aquifer, of unit cross section, releases from or gain to storage under a unit decline or rise in head.

$T = Kb$ = Transmissivity of the aquifer.

Equation for flow in horizontal leaky semiconfined aquifers of uniform thickness

In deriving the equation for nonleaky aquifers we had

$$\left.\frac{\partial h}{\partial z}\right|_b = -\frac{v_z}{K} = 0$$



In this case, from Darcy's law

$$\left.v_z\right|_b = K' \left.\frac{h|_b - h_1}{b'}\right|_b = K' \frac{\bar{h} - h_1}{b'} \text{ and } -\left.\frac{v_z}{K}\right|_b = -\frac{K'}{K}\frac{h_1 - \bar{h}}{b'}$$

Therefore

$$\frac{\partial^2}{\partial x^2}(b\bar{h}) + \frac{\partial^2}{\partial y^2}(b\bar{h}) + \frac{K'}{Kb}\frac{h_1 - \bar{h}}{b'} = \frac{S_s}{K}\frac{\partial}{\partial t}(b\bar{h})$$

[redacted] Dividing by b

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} + \frac{K'}{Kb} \cdot \frac{h_1 - \bar{h}}{b'} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t}$$

or

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} + \frac{h_1 - \bar{h}}{B^2} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t} = \frac{S}{T}\frac{\partial \bar{h}}{\partial t}$$

where

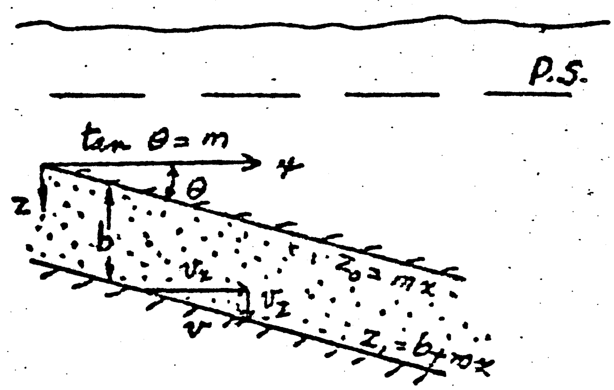
$B = \sqrt{T/K'/b'}$ = Leakage factor

and

K'/b' = Coefficient of leakage = Volume of water that crosses a unit cross-section of the semiconfined - confined bed interface, under a unit head difference.

<u>Equation for flow in inclined confined aquifers of uniform thickness</u>

Using the same method of averaging as before.



$$\int_{z_o}^{z_1} \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\, dz = \frac{S_s}{K}\int_{z_o}^{z_1} \frac{\partial h}{\partial t}\, dz$$

$$\int_{mx}^{b+mx} \frac{\partial^2 h}{\partial x^2}\, dz + \int_{mx}^{b+mx} \frac{\partial^2 h}{\partial y^2}\, dz + \int_{mx}^{b+mx} \frac{\partial^2 h}{\partial z^2}\, dz = \frac{S_s}{K}\int_{mx}^{b+mx} \frac{\partial h}{\partial t}\, dz$$

$$\int_{mx}^{b+mx} \frac{\partial^2 h}{\partial x^2}\, dz = \frac{\partial}{\partial x}\int_{mx}^{b+mx} \frac{\partial h}{\partial\partial x}\, dz + m\frac{\partial h}{\partial x}\bigg|_{mx} - m\frac{\partial h}{\partial x}\bigg|_{b+mx}$$

$$\int_{mx}^{b+mx} \frac{\partial^2 h}{\partial y^2}\, dz = \frac{\partial}{\partial y}\int_{mx}^{b+mx} \frac{\partial h}{\partial y}\, dz + \frac{\partial h}{\partial y}\frac{\partial(mx)}{\partial y} - \frac{\partial h}{\partial y}\frac{\partial(b+mx)}{\partial y} = \frac{\partial}{\partial y}\int_{mx}^{b+mx} \frac{\partial h}{\partial y}\, dz$$

$$\int_{mx}^{b+mx} \frac{\partial^2 h}{\partial z^2}\, dz = \frac{\partial h}{\partial z}\bigg|_{b+mx} - \frac{\partial h}{\partial z}\bigg|_{mx}$$

$$\int_{mx}^{b+mx} \frac{\partial h}{\partial t} dz = \frac{\partial}{\partial t}\int_{mx}^{b+mx} h dz$$

Therefore

$$\frac{\partial}{\partial x}\int_{mx}^{b+mx} \frac{\partial h}{\partial x} dz + \frac{mv_x}{K}\bigg|_{b+mx} - \frac{mv_x}{K}\bigg|_{mx} + \frac{\partial}{\partial y}\int_{mx}^{b+mx} \frac{\partial h}{\partial y} dz - \frac{v_z}{K}\bigg|_{b+mx} + \frac{v_z}{K}\bigg|_{mx} =$$

$$= \frac{S_s}{K}\frac{\partial}{\partial t}\int_{mx}^{b+mx} h dz$$

But

$\therefore$ $v_z = mv_x$ both at $z = b+mx$ and $z = mx$

$$\therefore \quad \frac{\partial}{\partial x}\int_{mx}^{b+mx} \frac{\partial h}{\partial x} dz + \frac{\partial}{\partial y}\int_{mx}^{b+mx} \frac{\partial h}{\partial y} dz = \frac{S_s}{K}\frac{\partial (b\bar{h})}{\partial t}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\int_{mx}^{b+mx} h dz + mh\bigg|_{mx} - mh\bigg|_{b+mx}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\int_{mx}^{b+mx} h dz\right) = \frac{S_s}{K}\frac{\partial (b\bar{h})}{\partial t}$$

$$\frac{\partial^2 (b\bar{h})}{\partial x^2} + m\frac{\partial h}{\partial x}\bigg|_{mx} - m\frac{\partial h}{\partial x}\bigg|_{b+mx} + \frac{\partial^2 (b\bar{h})}{\partial y^2} = \frac{S_s}{K}\frac{\partial (b\bar{h})}{\partial t}$$

Dividing by b, since it is a constant

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{m}{b}\frac{\partial h}{\partial x}\bigg|_{mx} - \frac{m}{b}\frac{\partial h}{\partial x}\bigg|_{b+mx} + \frac{\partial^2 \bar{h}}{\partial y^2} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t}$$

For small slopes $\frac{m}{b}$ will be a very small number; furthermore

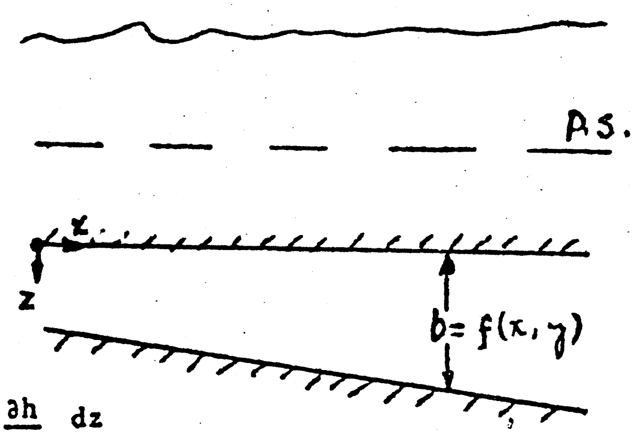
$$\left(\frac{\partial h}{\partial x}\bigg|_{mx} - \frac{\partial h}{\partial x}\bigg|_{b+mx}\right)$$

will also be very small and they therefore can be neglected. The resulting equation

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t} = \frac{S}{T}\frac{\partial \bar{h}}{\partial t}$$

is the same as that for horizontal aquifers.

Equation for flow in confined aquifers of non uniform thickness



b = f(x,y), i.e. the thickness is changing both in the x and y-direction. Proceeding as before to average

$$\int_{0}^{f(x,y)}\left(\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right)dz = \frac{S_s}{K}\int_0^{f(x,y)}\frac{\partial h}{\partial t}\,dz$$

from which, the final equation obtained will be

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} + \frac{1}{b}\frac{\partial b}{\partial x}\frac{\partial \bar{h}}{\partial x} + \frac{1}{b}\frac{\partial b}{\partial y}\frac{\partial \bar{h}}{\partial y} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t} = \frac{S}{T}\frac{\partial \bar{h}}{\partial t}$$

If the aquifer has a thickness changing only in the x-direction, then

$$\frac{\partial b}{\partial y} = 0$$

and consequently

$$\frac{\partial^2 \bar{h}}{\partial x^2} + \frac{\partial^2 \bar{h}}{\partial y^2} + \frac{1}{b}\frac{\partial b}{\partial x}\frac{\partial \bar{h}}{\partial x} = \frac{S_s}{K}\frac{\partial \bar{h}}{\partial t} = \frac{S}{T}\frac{\partial \bar{h}}{\partial t}$$

Condition on the free surface of an unconfined aquifer

In any point of an aquifer, the potential $\phi(x,y,z,t) = \frac{p}{\gamma} + z$

Taking total derivatives

$$\frac{D}{Dt}[\phi(x,y,z,t)] = \frac{D}{Dt}\left(\frac{p}{\gamma} + z\right)$$

where

$$\frac{D}{Dt}[f(x,y,z,t)] = \vec{v}_a \cdot \text{grad } f + \frac{\partial f}{\partial t}$$

$\vec{v}_a$ = the actual velocity vector

$$\therefore \quad \frac{\partial \phi}{\partial x} v_{x(a)} + \frac{\partial \phi}{\partial y} v_{y(a)} + \frac{\partial \phi}{\partial z} v_{z(a)} + \frac{\partial \phi}{\partial t} = v_{z(a)} + \frac{D}{Dt}(p/\gamma)$$

$$\left[\frac{D}{Dt}(z) = v_{z(a)} \text{ because } \vec{v}_a \text{ grad } z = v_{z(a)}\right]$$

and

$$v_{z(a)} = \frac{v_z}{S_y}, \quad v_{x(a)} = \frac{v_x}{S_y}, \quad v_{y(a)} = \frac{v_y}{S_y}$$

where

S_y = Specific yield = Amount of water that a unit volume of the aquifer will release from storage under the effect of gravity. (Numerically equal to effective porosity)

$$\therefore \quad \frac{\partial\phi}{\partial x}\frac{v_x}{S_y} + \frac{\partial\phi}{\partial y}\frac{v_y}{S_y} + \frac{\partial\phi}{\partial z}\frac{v_z}{S_y} + \frac{\partial\phi}{\partial t} = \frac{v_z}{S_y} + \frac{D}{Dt}(p/\gamma)$$

or

$$-K\left(\frac{\partial\phi}{\partial x}\right)^2 - K\left(\frac{\partial\phi}{\partial y}\right)^2 - K\left(\frac{\partial\phi}{\partial z}\right)^2 + S_y\frac{\partial\phi}{\partial t} = -K\frac{\partial\phi}{\partial z} + S_y\frac{D}{Dt}(p/\gamma)$$

Now, for an unconfined aquifer at $z = h$, $\phi = \frac{p_a}{\gamma} + h$,

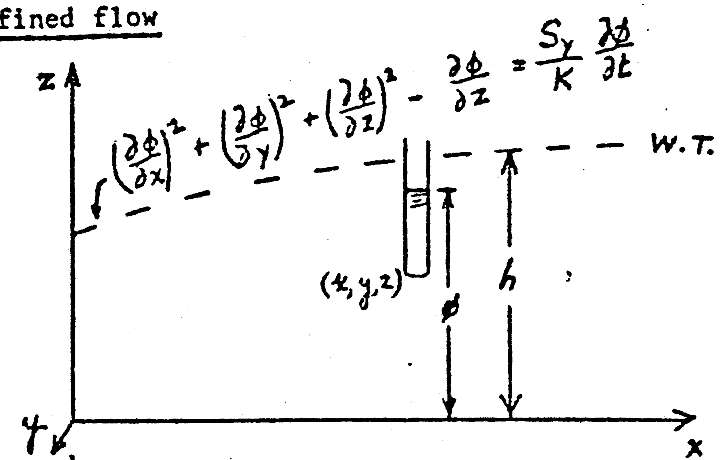
$$\therefore \quad \frac{\partial\phi}{\partial s} = \frac{\partial h}{\partial s} \text{ and } \frac{D}{Dt}(p_a/\gamma) = 0$$

$$\therefore \quad \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2 - \frac{\partial h}{\partial z} = \frac{S_y}{K}\frac{\partial h}{\partial t}$$

<u>Approximate Diff. Equation for unconfined flow</u>

Specific Storage

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{S_s}{K}\frac{\partial\phi}{\partial t}$$



Summing up the potentials in the z-direction

$$\int_0^h \frac{\partial^2\phi}{\partial x^2}dz + \int_0^h \frac{\partial^2\phi}{\partial y^2}dz + \int_0^h \frac{\partial^2\phi}{\partial z^2}dz = \frac{S_s}{K}\int_0^h \frac{\partial\phi}{\partial t}dz$$

$$\int_0^h \frac{\partial}{\partial x} v_x\, dz + \int_0^h \frac{\partial}{\partial y} v_y\, dz + \int_0^h \frac{\partial}{\partial z} v_z\, dz = - S_s \int_0^h \frac{\partial \phi}{\partial t}\, dz$$

$$\frac{\partial}{\partial x}\int_0^h v_x\, dz - v_x\Big|_h \frac{\partial h}{\partial x} + \frac{\partial}{\partial y}\int_0^h v_y\, dz - v_y\Big|_h \frac{\partial h}{\partial y} + v_z\Big|_h - v_z\Big|_0 =$$

$$= - S_s\left[\frac{\partial}{\partial t}\int_0^h \phi dz - \phi\Big|_h \frac{\partial h}{\partial t}\right]$$

Using the condition of the free surface after solving it for $\left.\frac{\partial \phi}{\partial z}\right|_h = - \left.\frac{v_z}{K}\right|_h$

$$\frac{\partial}{\partial x}\int_0^h v_x dz + K\left(\frac{\partial h}{\partial x}\right)^2 + \frac{\partial}{\partial y}\int_0^h v_y dz + K\left(\frac{\partial h}{\partial y}\right)^2$$

$$+\left[S_y \frac{\partial h}{\partial t} - K\left(\frac{\partial h}{\partial x}\right)^2 - K\left(\frac{\partial h}{\partial y}\right)^2 - K\left(\frac{\partial h}{\partial z}\right)^2\right] = - S_s\left[\frac{\partial}{\partial t}\int_0^h \phi dz - h\frac{\partial h}{\partial t}\right]$$

$$\frac{\partial}{\partial x}\int_0^h v_x dz + \frac{\partial}{\partial y}\int_0^h v_y dz - K\left(\frac{\partial h}{\partial z}\right)^2 = - S_y \frac{\partial h}{\partial t} - S_s \frac{\partial}{\partial t}\int_0^h \phi dz - h\frac{\partial h}{\partial t} \quad \text{———(A)}$$

With the following assumptions:

1) The gradient is small compared to the total thickness and $\therefore \left(\frac{\partial h}{\partial z}\right)^2 = 0$

2) $S_s << S_y$ and $\therefore S_s = 0$ the resulting expression will be

$$\frac{\partial}{\partial x}(h\bar{v}_x) + \frac{\partial}{\partial y}(h\bar{v}_y) = - S_y \frac{\partial h}{\partial t}$$

Replacing $\bar{v}_s = -K \frac{\partial \bar{h}}{\partial s}$ ($\bar{h}$ = potential producing $\bar{v}_s$)

$$\frac{\partial}{\partial x} h \frac{\partial \bar{h}}{\partial x} + \frac{\partial}{\partial y} h \frac{\partial \bar{h}}{\partial y} = \frac{S_Y}{K} \frac{\partial h}{\partial t}$$

Assuming $h \approx \bar{h}$ at the surface

$$\frac{\partial^2}{\partial x^2}\left(\frac{\bar{h}^2}{2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\bar{h}^2}{2}\right) = \frac{S_Y}{K} \frac{\partial \bar{h}}{\partial t}$$

Multiplying and dividing the right hand side by $\bar{h}$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\bar{h}^2}{2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\bar{h}^2}{2}\right) = \frac{S_Y}{K\bar{h}} \frac{\partial}{\partial t}\left(\frac{\bar{h}^2}{2}\right)$$

or

$$\frac{\partial^2(\bar{h}^2)}{\partial x^2} + \frac{\partial^2(\bar{h}^2)}{\partial y^2} = \frac{S_Y}{K\bar{D}} \frac{\partial}{\partial t}(\bar{h}^2)$$

where

$$\bar{D} = \bar{h} = \text{average depth} = \frac{h_1 + h_2}{2}$$

The equation is linear in $\bar{h}^2$ and can be solved for $\bar{h}^2$.

Another Diff. Equation for unconfined flow involving the average potential $\bar{\phi}$.

Another equation for unconfined flow can be derived, by a different procedure, starting from the equation (A) on page 22.

$$\frac{\partial}{\partial x}\int_0^h v_x dz + \frac{\partial}{\partial y}\int_0^h v_y dz - K\left(\frac{\partial h}{\partial z}\right)^2 = S_y \frac{\partial h}{\partial t} - S_s\left[\frac{\partial}{\partial t}\int_0^h \phi dz - h\frac{\partial h}{\partial t}\right]$$

Instead of averaging the velocities as before, the potentials will be averaged.

$$\frac{\partial}{\partial x}\left[-\int_0^h K\frac{\partial \phi}{\partial x}dz\right] + \frac{\partial}{\partial y}\left[-\int_0^h K\frac{\partial \phi}{\partial y}dz\right] - K\left(\frac{\partial h}{\partial z}\right)^2 = -S_y\frac{\partial h}{\partial t} - S_s\left[\frac{\partial}{\partial t}\left(h\bar{\phi} - \frac{h^2}{2}\right)\right]$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\int_0^h \phi dz - h\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\int_0^h \phi dz - h\frac{\partial h}{\partial y}\right] + \left(\frac{\partial h}{\partial z}\right)^2 = \frac{S_y}{K}\frac{\partial h}{\partial t} + \frac{S_s}{K}\frac{\partial}{\partial t}\left(h\bar{\phi} - \frac{h^2}{2}\right)$$

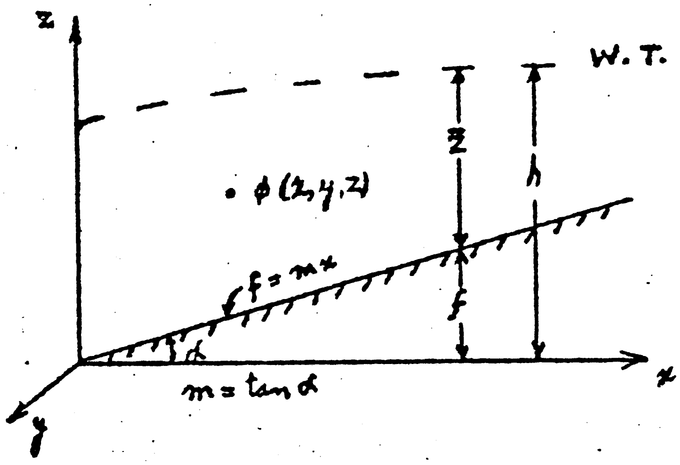
$$\frac{\partial^2}{\partial x^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) + \frac{\partial^2}{\partial y^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) + \left(\frac{\partial h}{\partial z}\right)^2 = \frac{S_y}{K}\frac{\partial h}{\partial t} + \frac{S_s}{K}\frac{\partial}{\partial t}\left(h\bar{\phi} - \frac{h^2}{2}\right)$$

This is a rigorous equation for unconfined flow. As it is very complicated and difficult to solve it can be approximated by making the same assumptions as before (bottom of page 22).

$$\frac{\partial^2}{\partial x^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) + \frac{\partial^2}{\partial y^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) = \frac{S_y}{K}\frac{\partial h}{\partial t}$$

Approximate Diff. Equation for inclined unconfined aquifers

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{S_s}{K}\frac{\partial\phi}{\partial t}$$



we can also write

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = -S_s\frac{\partial\phi}{\partial t}$$

The specific storage in an unconfined aquifer is very small, $S_s \approx 0$. Averaging the remaining part of the equation

$$\int_f^h \frac{\partial v_x}{\partial x}dz + \int_f^h \frac{\partial v_y}{\partial y}dz + \int_f^h \frac{\partial v_z}{\partial z}dz = 0$$

$$\frac{\partial}{\partial x}\int_f^h v_x dz - v_x\Big|_h \frac{\partial h}{\partial x} + v_x\Big|_f \frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\int_f^h v_y dz - v_y\Big|_h \frac{\partial h}{\partial y} + v_y\Big|_f \frac{\partial f}{\partial y} + v_z\Big|_h - v_z\Big|_f = 0$$

$$\frac{\partial}{\partial x}\Big[(h-f)\bar{v}_x\Big] + \frac{\partial}{\partial y}\Big[(h-f)\bar{v}_y\Big] + K\left(\frac{\partial h}{\partial x}\right)^2 + v_x\Big|_f \frac{\partial f}{\partial x} + K\left(\frac{\partial h}{\partial y}\right)^2 + v_y\Big|_f \frac{\partial f}{\partial y} - v_z\Big|_f - K\frac{\partial h}{\partial z} = 0$$

With the following steps:

1) Substitute $K\frac{\partial h}{\partial z}$ with its value from the condition on the free surface
2) Replace $\frac{\partial f}{\partial x} = m$, $\frac{\partial f}{\partial y} = 0$ and $h-f = \bar{z}$
3) Neglect second degree differential $\left(\frac{\partial h}{\partial z}\right)^2$
4) Note that $\bar{v}_s = -K\frac{\partial\bar{\phi}}{\partial s}$ and $mv_x\Big|_f = v_z\Big|_f$

Also $\bar{\phi} \approx h = \bar{z} + f = \bar{z} + mx$

we obtain

$$\frac{\partial}{\partial x} \bar{z} \frac{\partial}{\partial x} (\bar{z}+mx) + \frac{\partial}{\partial y} \bar{z} \frac{\partial}{\partial y} (\bar{z}+mx) = \frac{S_y}{K} \frac{\partial}{\partial t} (\bar{z}+mx)$$

$$\frac{\partial}{\partial x} \bar{z} \frac{\partial \bar{z}}{\partial x} + \bar{z}m + \frac{\partial}{\partial y} \bar{z} \frac{\partial \bar{z}}{\partial y} = \frac{S_y}{K} \frac{\partial \bar{z}}{\partial t}$$

$$\frac{1}{2} \frac{\partial^2 \bar{z}^2}{\partial x^2} + m \frac{\partial \bar{z}}{\partial x} + \frac{1}{2} \frac{\partial^2 \bar{z}^2}{\partial y^2} = \frac{S_y \bar{z}}{K\bar{z}} \frac{\partial \bar{z}}{\partial t}$$

$$\frac{\partial^2 \bar{z}^2}{\partial x^2} + \frac{\partial^2 \bar{z}^2}{\partial y^2} + 2 \frac{m\bar{z}}{\bar{z}} \frac{\partial \bar{z}}{\partial x} = \frac{S_y}{K\bar{z}} \frac{\partial \bar{z}^2}{\partial t}$$

$$\frac{\partial^2 \bar{z}^2}{\partial x^2} + \frac{\partial^2 \bar{z}^2}{\partial y^2} + \frac{m}{\bar{z}} \frac{\partial \bar{z}^2}{\partial x} = \frac{S_y}{K\bar{z}} \frac{\partial \bar{z}^2}{\partial t}$$

Replacing $\bar{z}$ by $\bar{D}$ = average depth, we finally obtain

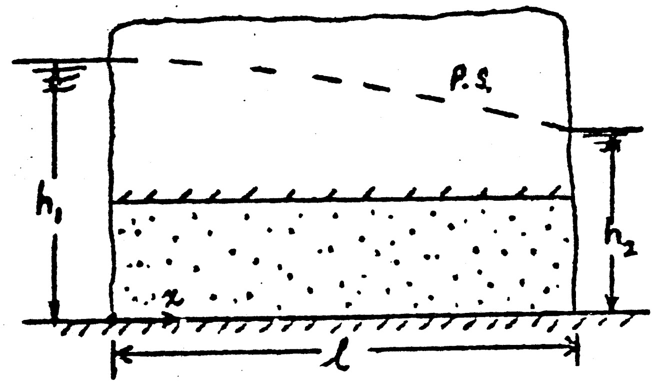
$$\frac{\partial^2 \bar{z}^2}{\partial x^2} + \frac{\partial^2 \bar{z}^2}{\partial y^2} + \frac{m}{\bar{D}} \frac{\partial \bar{z}^2}{\partial x} = \frac{S_y}{K\bar{D}} \frac{\partial \bar{z}^2}{\partial t}$$

It should be kept in mind that $\bar{z}$ is the elevation of the water table above the impermeable bed and not above the x-axis.

STEADY FLOW PROBLEMS

Flow between two line sources in a confined aquifer:

$$\frac{\partial^2 h}{\partial x^2} = 0 \quad (1)$$



$$h(0) = h_1 \quad (2)$$

$$H(\ell) = h_2 \quad (3)$$

From (1) $\quad h = c_1 x + c_2$

From (2) $\quad h_1 = c_2$

From (3) $\quad c_1 = -\frac{h_1 - h_2}{\ell}$

$$\therefore \quad h = h_1 - \frac{h_1 - h_2}{\ell} x$$

The intensity of flow:

$$q = vA$$

$$= b \times 1 \left(-K \frac{\partial h}{\partial x}\right) = -T \frac{\partial h}{\partial x}$$

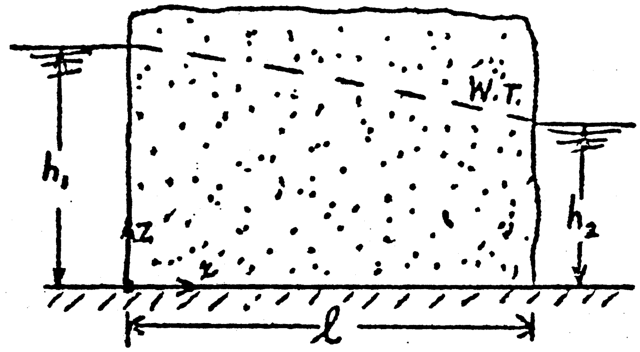
$$\frac{\partial h}{\partial x} = -\frac{h_1 - h_2}{\ell}$$

and

$$q = T \frac{h_1 - h_2}{\ell}$$

Flow between two line sources in an unconfined aquifer

$$\frac{\partial^2 z^2}{\partial x^2} = 0 \qquad (1)$$



$$z(0) = h_1 \qquad (2)$$

$$z(\ell) = h_2 \qquad (3)$$

From (1) $\qquad z^2 = c_1 x + c_2$

From (2) $\qquad c_2 = h_1^2$

From (3) $\qquad c_1 = -\frac{(h_1^2 - h_2^2)}{\ell}$

and $\qquad z^2 = h_1^2 - \frac{h_1^2 - h_2^2}{\ell} x$

or. $\qquad h_1^2 - z^2 = \frac{h_1^2 - h_2^2}{\ell} x$

The intensity of flow:

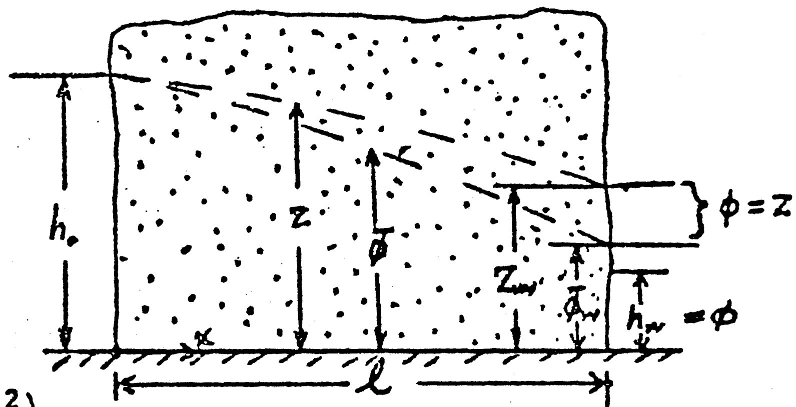
$$q = Av = - Kz \frac{\partial z}{\partial x}$$

$$z \frac{\partial z}{\partial x} = - \frac{h_1^2 - h_2^2}{2\ell}$$

$$q = \frac{K}{2\ell} (h_1^2 - h_2^2)$$

<u>Flow between two line sources in an unconfined aquifer</u>

The problem solved on page 28, will this time be solved by using the equation containing the average potential $\bar{\phi}$.



$$\frac{\partial^2}{\partial x^2}\left(z\bar{\phi} - \frac{z^2}{2}\right) = 0 \qquad (1)$$

$$z(0) = h_o \qquad (2)$$

$$z(\ell) = z_w \qquad (3)$$

$$\bar{\phi}(0) = h_o \qquad (4)$$

$$\bar{\phi}(\ell) = \bar{\phi}_w = \frac{\int_0^{z_w} \phi(\ell,z)dz}{z_w}$$

$$= \frac{1}{z_w}\left[\int_0^{h_w} h_w dz + \int_{h_w}^{z_w} zdz\right]$$

$$= \frac{h_w^2 + z_w^2}{2z_w} \qquad (5)$$

From (1) $\qquad z\bar{\phi} - \frac{z^2}{2} = c_1 x + c_2$

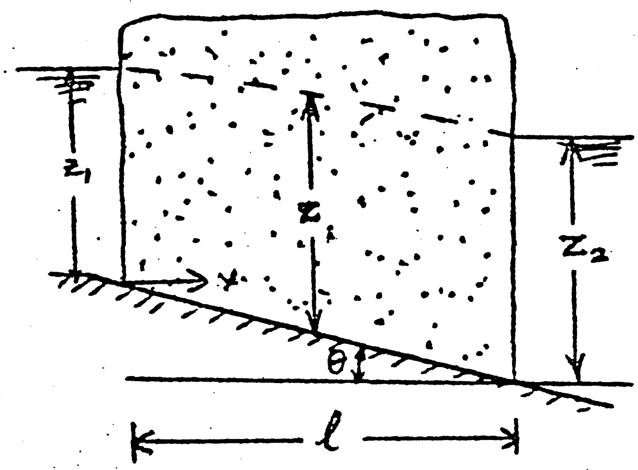
From (2) & (4) $\qquad c_2 = \frac{h_0^2}{2}$

From (3) & (5) $$c_1 = -\frac{h_0^2 - h_w^2}{2\ell}$$

and $$z\bar{\phi} - \frac{z^2}{2} = \frac{h_0^2}{2} - \frac{h_0^2 - h_w^2}{2\ell}\,x$$

<u>Flow between two line sources in an unconfined inclined aquifer</u>

Noting that the slope $m = \tan\theta$ is negative



$$\frac{\partial^2 z^2}{\partial x^2} - \frac{m}{\bar{D}}\frac{\partial z^2}{\partial x} = 0 \qquad (1)$$

$$z(0) = z_1 \qquad (2)$$

$$z(\ell) = z_2 \qquad (3)$$

From (1) $$z^2 = c_1 e^{\frac{m}{\bar{D}}x} + c_2$$

From (2) $$z_1^2 = c_1 + c_2$$

From (3) $$z_2^2 = c_1 e^{\frac{m}{\bar{D}}\ell} = c_2$$

After solving for c_1 and c_2

$$z_1^2 - z^2 = \frac{z_1^2 - z_2^2}{\left(1 - e^{\frac{m\ell}{\bar{D}}}\right)}\left(1 - e^{\frac{m}{\bar{D}}x}\right)$$

or

$$\frac{z_1^2 - z_2^2}{z_1^2 - z_2^2} = e^{-\frac{m}{2\bar{D}}(\ell - x)}\,\frac{\sinh\frac{mx}{2\bar{D}}}{\sinh\frac{m\ell}{2\bar{D}}}$$

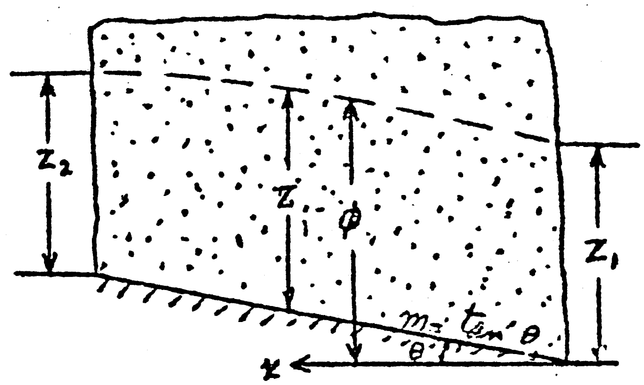
The intensity of flow at $x = 0$ & $z = z_1$

$$q = Av$$

$$= z.1\left(-K \left.\frac{\partial \phi}{\partial x}\right|_{o}\right) = -Kz\left(\frac{\partial z}{\partial x} - m\right)_{0,z_1}$$

$$q = \frac{K(z_1^2 - z_2^2)}{2 \sinh \frac{m \ell}{2\bar{D}}} \quad \frac{m}{2\bar{D}} \quad e^{-\frac{m\ell}{2\bar{D}}} + Kmz_1$$

<u>Flow between two line sources in an unconfined inclined aquifer</u>



The problem was solved on page 30 by using the equation

$$\frac{\partial^2 z^2}{\partial x^2} + \frac{m}{\bar{D}}\frac{\partial z^2}{\partial x} = 0$$

which involves the approximation

$$z = \bar{D} = \frac{z_1 + z_2}{2}$$

It will now be solved without this approximation.

$$\frac{\partial^2 z^2}{\partial x^2} + \frac{m}{z}\frac{\partial z^2}{\partial x} = 0$$

Noting that

$$\frac{\partial}{\partial x}(z^2) = 2z\,\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}\left(2z\,\frac{\partial z}{\partial x}\right) + 2m\;\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\left(z\,\frac{\partial z}{\partial x} + mz\right) = 0$$

$$z\left(\frac{\partial z}{\partial x} + m\right) = c$$

But

$$q_x = \left(K\,\frac{\partial \phi}{\partial x}\right) z \cdot 1 \qquad \text{and} \qquad \phi = z + mx$$

$$\therefore\; q_x = Kz\left(\frac{\partial z}{\partial x} + m\right)$$

and

$$c = \frac{q_x}{K}$$

$$Kz\,\frac{\partial z}{\partial x} + Kmz = q_x$$

$$\frac{Kz\,\partial z}{q_x - Kmz} = \partial x$$

$$\partial x = \frac{1}{m}\left(-1 + \frac{q_x}{q_x - Kmz}\right)\partial z$$

$$x = -\frac{q_x}{Km^2}\ln\left(q_x - Kmz\right) - \frac{z}{m} + c$$

@ $x = 0 \quad z = z_1$

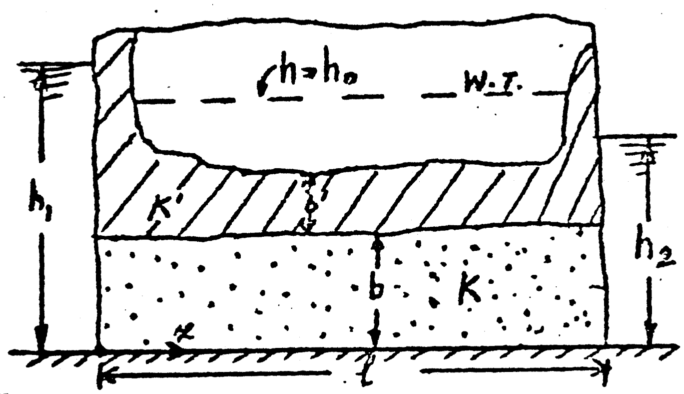
$$c = \frac{q_x}{Km^2} \ln (q_x - Kmz_1) + \frac{z_1}{m}$$

and

$$mx = \frac{q_x}{Km} \ln \frac{(q_x/Km) - z_1}{(q_x/Km) - z} + (z_1 - z)$$

Note: q_x is constant since the flow is steady. The subscript x is used to denote that the discharge is in the x direction and it can be omitted.

Flow between two line sources in a leaky aquifer



$$\frac{\partial^2 h}{\partial x^2} + \frac{h_o - h}{B^2} = 0 \qquad (1)$$

$$h(0) = h_1 \qquad (2)$$

$$h(\ell) = h_2 \qquad (3)$$

From (1) $\quad h = c_1 \sinh \frac{x}{B} + c_2 \cosh \frac{x}{B} + h_o$

From (2) $\quad c_2 = h_1 - h_o$

From (3) $\quad c_1 = \dfrac{h_2 - h_o - (h_1 - h_o) \cosh \ell/B}{\sinh \ell/B}$

$$h = h_o + \frac{h_2 - h_o - (h_1-h_o) \cosh \ell/B}{\sinh \ell/B} \sinh \frac{x}{B} + (h_1-h_o)\cosh \frac{x}{B}$$

$$= h_o + \frac{h_2 - h_o}{\sinh \ell/B} \sinh \frac{x}{B} + \frac{h_1 - h_o}{\sinh \ell/B} \left[\sinh \frac{\ell}{B} \cosh \frac{x}{B} - \cosh \frac{\ell}{B} \sinh \frac{x}{B}\right]$$

or

$$h = h_o + \frac{h_2 - h_o}{\sinh \ell/B} \sinh \frac{x}{B} + \frac{h_1 - h_o}{\sinh \ell/B} \sinh \frac{\ell - x}{B}$$

The intensity of flow:

$$q = Av = -T \frac{\partial h}{\partial x}$$

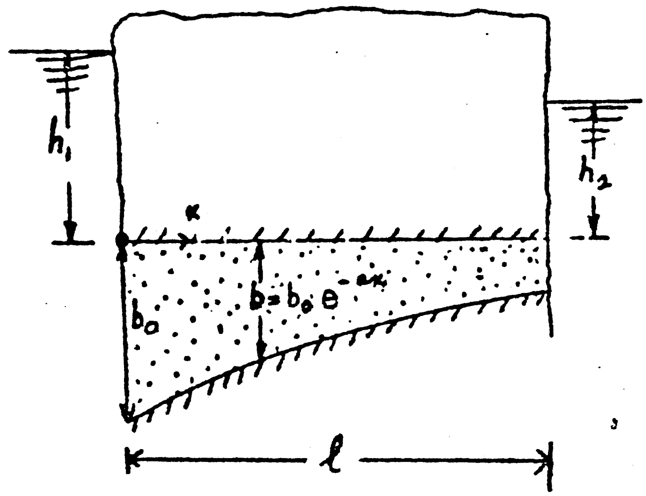
$$q = \frac{T}{B \sinh \ell/B} \left[(h_1 - h_o) \cosh \frac{\ell - x}{B} - (h_2 - h_o) \cosh \frac{x}{B} \right]$$

Flow between two line sources in a confined aquifer of non-uniform thickness

$$\frac{\partial^2 h}{\partial x^2} + \frac{1}{b} \frac{\partial b}{\partial x} \frac{\partial h}{\partial x} = 0 \qquad (1)$$

$$h(0) = h_1 \qquad (2)$$

$$h(\ell) = h_2 \qquad (3)$$



$$b = b_o e^{-ax}$$

$$\frac{\partial b}{\partial x} = -a b_o e^{-ax}$$

(1) reduces to $$\frac{\partial^2 h}{\partial x^2} - a \frac{\partial h}{\partial x} = 0$$

$$h = c_1 e^{+ax} + c_2$$

From (2) $$h_1 = c_1 + c_2$$

From (3) $$h_2 = c_1 e^{+a\ell} + c_2$$

After solving for c_1 & c_2

$$h = h_1 - \frac{h_1 - h_2}{(1 - e^{+a\ell})}(1 - e^{+ax})$$

Intensity of flow

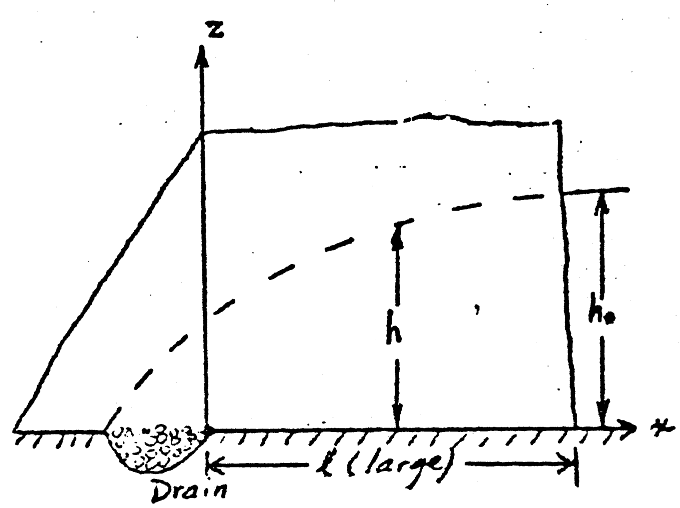
$$q = -T_b \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial x} = -a\left(\frac{h_1 - h_2}{1 - e^{-a\ell}}\right)e^{+ax}$$

$$q = T_b a\left(\frac{h_1 - h_2}{1 - e^{-a\ell}}\right)e^{+ax}$$

$$T_b = Kb = Kb_0 e^{-ax}$$

<u>Seepage through a thick earth dam</u>



This problem is taken as an example to show that the equation

$$\frac{\partial^2}{\partial x^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) = 0$$

gives very rigorous results for the height of the water table, although it contains the term of average potential $\bar{\phi}$.

The solution to the general diff. equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

applying to this case can be obtained by using complex variables or other advanced methods. The solution which satisfies all the boundary conditions is

$$\phi(x,z) = \sqrt{\frac{q}{K}}\left[x + \sqrt{x^2 + z^2}\right]^{1/2}$$

By taking the appropriate derivatives, it can be easily shown that the general diff. equation and the condition on the free surface:

$$\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 - \frac{\partial\phi}{\partial z}\right]_{z=h} = 0$$

are satisfied.

For large values of ℓ, the equation is adequate to be used for the height of the water table. If ℓ is large compared to z, then

$$\phi(x,z) \approx \phi(x) = h$$

$$\phi(x,z) = \sqrt{\frac{q}{K}}\left[x + \sqrt{x^2 + z^2}\right]^{1/2}$$

at the water table $\qquad \phi = h = z$

$$h = \sqrt{\frac{q}{K}}\ [x + \sqrt{x^2 + h^2}]^{1/2}$$

$$h^2 = 2\,\frac{q}{K}\,x + \left(\frac{q}{K}\right)^2$$

which is the equation for the water table.

@ $x = 0$ $h(0) = \frac{q}{K}$

$$\bar{\phi}(0) = \frac{\int_0^{h(0)} \phi(0,z)\ dz}{h(0)}$$

$$= \frac{\sqrt{q/K}}{h(0)} \int_o^{h_o} z^{1/2}\ dz$$

$$= \frac{2}{3}\sqrt{\frac{q}{K}}\ \sqrt{h(0)} = \frac{2}{3}\ \frac{q}{K}$$

Base hydraulic head

$$\phi(x,0) = \sqrt{\frac{q}{K}}\ [x + |x|]^{1/2}$$

for $x \geq 0$

$$\phi(x,0) = \sqrt{2\,\frac{q}{K}\,x}$$

for $x \leq 0$

$$\phi(x,0) = 0$$

The width of the drainage ditch

$$0 = 2\frac{q}{K}x + \left(\frac{q}{K}\right)^2$$

$$x = -\frac{q}{2K}$$

To find the discharge at $x = \ell$, $h = h_o$

From the W.T. equation

$$\frac{q}{K} = \frac{-2\ell \pm \sqrt{4\ell^2 + 4h_o^2}}{2} \quad \text{(it has to be positive.)}$$

$$\ell = \left[\left(1 + \frac{h_o^2}{\ell^2}\right)^{1/2} - 1\right]$$

Noting that

$$(1 + \varepsilon)^{1/2} = 1 + \frac{\varepsilon}{2} + \ldots\ldots$$

$$\frac{q}{K} = \frac{h_o^2}{2\ell}$$

Now we will use the equation containing the $\bar{\phi}$ term to show that it leads to the same solution for the water table.

$$\frac{\partial^2}{\partial x^2}\left(h\bar{\phi} - \frac{h^2}{2}\right) = 0 \qquad (1)$$

$$\bar{\phi}(0) = \frac{2}{3}\frac{q}{K} \qquad (2)$$

$$h(0) = \frac{q}{K} \qquad (3)$$

$$q = K \int_0^h \frac{\partial \phi}{\partial x}\, dz$$

$$\frac{q}{K} = \frac{\partial}{\partial x} \int_0^h \phi(x,z)\ dz - \phi(x,h)\ \frac{\partial h}{\partial x}$$

$$= \frac{\partial}{\partial x} \left[\frac{h \int_0^h \phi(x,z)\ dz}{h}\right] - h\ \frac{\partial h}{\partial x}$$

$$= \frac{\partial}{\partial x}\ (h\bar{\phi}) - \frac{1}{2}\frac{\partial}{\partial x}\ h^2$$

$$= \frac{\partial}{\partial x}\ (h\bar{\phi} - \frac{h^2}{2}) \qquad (4)$$

From (1) $\qquad \frac{\partial}{\partial x}\ (h\bar{\phi} - \frac{h^2}{2}) = c$

From (4) $\qquad c = \frac{q}{K}$

$$\therefore\ h\bar{\phi} - \frac{h^2}{2} = \frac{q}{K}\, x + c_1$$

From (2) & (3) $\qquad c_1 = \frac{2}{3}\left(\frac{q}{K}\right)^2 - \frac{1}{2}\left(\frac{q}{K}\right)^2 = \frac{1}{6}\left(\frac{q}{K}\right)^2$

$$h\bar{\phi} - \frac{h^2}{2} = \frac{q}{K}x + \frac{1}{6}\left(\frac{q}{K}\right)^2$$

From the previous solution

$$\phi(x,z) = \sqrt{\frac{q}{K}}\,[x + \sqrt{x^2 + z^2}]^{1/2}$$

$$\therefore \bar{\phi} = \frac{\sqrt{\frac{q}{K}} \int_0^h [x + \sqrt{x^2 + z^2}]^{1/2}\,dz}{h}$$

Substituting $x^2 + z^2 = \beta^2$ $\qquad zdz = \beta d\beta$

Upper lim. : $\sqrt{x^2 + h^2}$ $\qquad$ Lower lim. : x

$$h\bar{\phi} = \sqrt{\frac{q}{K}} \int_x^{\sqrt{x^2+h^2}} \frac{\sqrt{x + \beta}}{\sqrt{\beta^2 - x^2}}\,\beta d\beta$$

$$= \sqrt{\frac{q}{K}} \int_x^{\sqrt{x^2+h^2}} \frac{\beta}{\sqrt{\beta - x}}\,d\beta$$

Substitute $\beta - x = u^2$ $\qquad d\beta = 2udu$

Upper lim = $[\sqrt{x^2 + h^2} - x]^{1/2}$ $\qquad$ Lower lim: 0

$$h\bar{\phi} = \sqrt{\frac{q}{K}} \int_0^{[\sqrt{x^2+h^2} - x]^{1/2}} 2(u^2 + x) \, du$$

$$= \frac{2}{3}\sqrt{\frac{q}{K}} [3x + \sqrt{x^2+h^2} - x] \, [\sqrt{x^2 + h^2} - x]^{1/2}$$

From the equation of the W.T.

$$h^2 = 2\frac{q}{K}x + \left(\frac{q}{K}\right)^2$$

or

$$x^2 + h^2 = \left(\frac{q}{K} + x\right)^2$$

and

$$h\bar{\phi} = 2\frac{q}{K}x + \frac{2}{3}\left(\frac{q}{K}\right)^2$$

$$h\bar{\phi} - \frac{h^2}{2} = \frac{q}{K}x + \frac{1}{6}\left(\frac{q}{K}\right)^2$$

Replacing the value of $h\bar{\phi}$ obtained

$$2\frac{q}{K}x + \frac{2}{3}\left(\frac{q}{K}\right)^2 - \frac{h^2}{2} = \frac{q}{K}x + \frac{1}{6}\left(\frac{q}{K}\right)^2$$

$$h^2 = 4\frac{q}{K}x + \frac{4}{3}\left(\frac{q}{K}\right)^2 - 2\frac{q}{K}x - \frac{1}{3}\left(\frac{q}{K}\right)^2$$

$$h^2 = 2\frac{q}{K}x + \left(\frac{q}{K}\right)^2$$

Which is the same equation as the one obtained before, and consequently rigorously valid.

Flow between three line sources in a confined aquifer

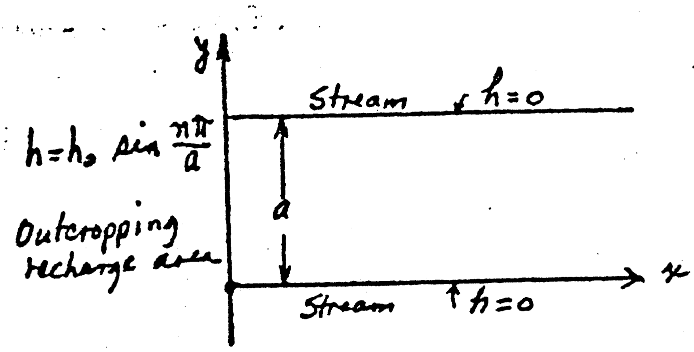
$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (1)$$

$$h(0,y) = h_o \sin \frac{\pi y}{a} \qquad (2)$$

$$h(x,0) = 0 \qquad (3)$$

$$h(x,a) = 0 \qquad (4)$$

$$h(\infty,y) = 0 \qquad (5)$$



Solving (1) by separation of variables (h = x·y) and satisfying (3) and (5)

$$h = Ae^{-\alpha x} \sin \alpha y$$

From (2)

$$h_o \sin \frac{\pi y}{a} = A \sin \alpha y$$

$$A = h_o \quad , \quad \alpha = \frac{\pi}{a}$$

and

$$h = h_o e^{-\frac{\pi x}{a}} \sin \frac{\pi y}{a}$$

Intensity of recharge

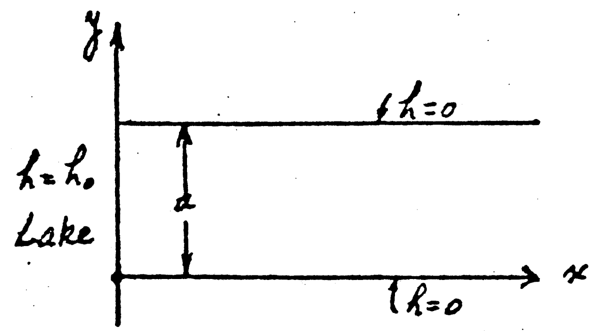
$$q = - T \left.\frac{\partial h}{\partial x}\right|_0$$

$$\frac{\partial h}{\partial x} = - \frac{h_o \pi}{a} e^{-\frac{\pi x}{a}} \sin \frac{\pi y}{a}$$

$$q = \frac{Th_o \pi}{a} \sin \frac{\pi y}{a}$$

<u>Flow between three line sources in a confined aquifer</u>

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (1)$$



$$h(0,y) = h_o \qquad (2)$$

$$h(x,0) = 0 \qquad (3)$$

$$h(x,a) = 0 \qquad (4)$$

$$h(\infty,y) = 0 \qquad (5)$$

Solving by separation of variables and satisfying conditions (3) and (5)

$$h = Ae^{-\alpha x} \sin \alpha y$$

From (4) $\qquad 0 = Ae^{-\alpha x} \sin \alpha a$

$$\alpha = \frac{n\pi}{a}$$

$$h = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}x} \sin \frac{n\pi y}{a}$$

From (2)

$$h_o = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y$$

Expanding h_o in a Fourier sine series

$$A_n = \frac{2h_o}{a}\int_0^a \sin\frac{n\pi y}{a}\,dy = \frac{2h_o}{a}\left[\frac{-\cos\frac{n\pi y}{a}}{\frac{n\pi}{a}}\right]_0^a = \frac{2h_o}{n\pi}\left[1-(-)^n\right]$$

and
$$h = \frac{2h_o}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-)^n}{n}\right] e^{-\frac{n\pi}{a}x}\sin\frac{n\pi}{a}y$$

or
$$h = \frac{4h_o}{\pi}\sum_{K=0}^{\infty}\frac{1}{2K+1}\, e^{-\frac{2K+1}{a}\pi x}\sin\frac{2K+1}{a}\pi y$$

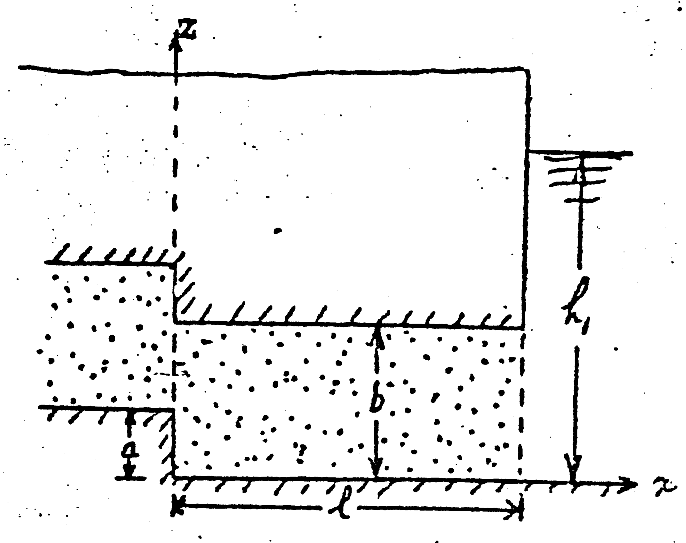
<u>Flow in a faulted confined aquifer</u>

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} = 0 \qquad (1)$$

$$h(\ell,z) = h_1 \qquad (2)$$

$$\frac{\partial h}{\partial z}(x,b) = 0 \qquad (3)$$

$$\frac{\partial h}{\partial z}(x,0) = 0 \qquad (4)$$



$$\frac{\partial h}{\partial x}(0,z) = \begin{cases} 0 & 0 < z < a \\ \frac{q}{K}\frac{1}{b-a} & a < z < b \end{cases} \qquad (5)$$

Solving (1) by separation of variables, combining the exponentials in hyperbolic function forms (more convenient for problems involving finite lengths) and with the help of the boundary conditions:

$$h = c + c_1 \sinh \alpha (\ell - x) \cos \alpha z$$

From (2) $\quad c = h_1$

From (3) $\quad \alpha = \frac{n\pi}{b}$

$$h = h_1 + \sum_{n=0}^{\infty} A_n \sinh \frac{n\pi}{b} (\ell - x) \cos \frac{n\pi}{b} z$$

From (5)

$$\left.\frac{\partial h}{\partial x}\right|_{x=0} = \sum_{n=0}^{\infty} A_n \left(-\frac{n\pi}{b}\right) \cosh \frac{n\pi\ell}{b} \cos \frac{n\pi z}{b}$$

or

$$f(z) = \begin{cases} 0 & 0 < z < a \\ \frac{q}{K(b-a)} & a < z < b \end{cases} = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi z}{b}$$

where $\quad a_n = - A_n \frac{n\pi}{b} \cosh \frac{n\pi\ell}{b}$

Expanding $f(z)$ in a Fourier cosine series in the interval $0 < z < b$

$$a_o = \frac{2}{b} \int_0^z f(z)\, dz = \frac{2}{b} \int_a^b \frac{q}{K(b-a)}\, dz = \frac{2q}{Kb}$$

$$a_n = \frac{2}{b} \int_0^z f(z) \cos \frac{n\pi z}{b} \, dz$$

$$= \frac{2}{b} \int_a^b \frac{q}{K(b-a)} \cos \frac{n\pi z}{b} \, dz = - \frac{2q}{n\pi K(b-a)} \sin \frac{n\pi a}{b}$$

and

$$A_n = \frac{2qb}{n^2 \pi^2 K(b-a)} \cdot \frac{\sin \frac{n\pi a}{b}}{\cosh \frac{n\pi \ell}{b}}$$

$$h = h_1 + \frac{a_o}{2} x + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{b} (\ell - x) \cos \frac{n\pi z}{b}$$

But at $x = \ell$, $h = h_1$, therefore second term should be $\frac{a_o}{2}(x-\ell)$. Replacing the values of a_o and A_n, we finally obtain

$$h = h_\ell + \frac{q}{Kb}(x-\ell)$$

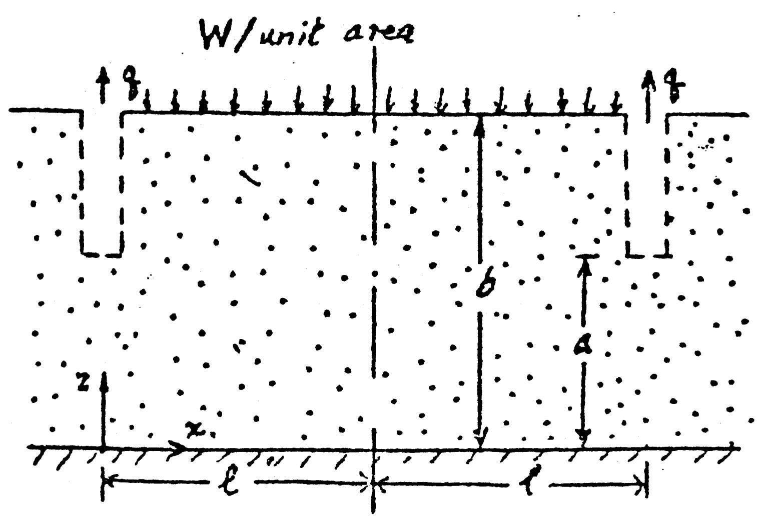
$$+ \frac{2qb}{\pi^2 K(b-a)} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi a}{b}}{n^2 \cosh \frac{n\pi \ell}{b}} \cdot \sinh \frac{n\pi}{b} (\ell - x) \cos \frac{n\pi z}{b}$$

<u>Flow to a series of partially penetrating open drains in balance with a uniform rate of accretion</u>

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (1)$$

$$\frac{\partial \phi}{\partial x}(2\ell, z) = \frac{\partial \phi}{\partial x}(0, z) = 0 \qquad (2)$$

for $0 < z < a$



$$\frac{\partial\phi}{\partial x}(2\ell,z) = \frac{\partial\phi}{\partial x}(0,z) = \frac{q}{2K(b-a)} \quad (3)$$

for $a < z < b$

$$\frac{\partial\phi}{\partial x}(\ell,z) = 0 \quad (4)$$

$$\frac{\partial\phi}{\partial z}(x,0) = 0 \quad (5)$$

$$\frac{\partial\phi}{\partial z}(x,b) = \frac{w}{K} \quad (6)$$

Solving (1) by separation of variables

$$\phi = c_1 x + c_2 (x^2 - z^2) + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi}{b}(\ell-x) \cos \frac{n\pi z}{b}$$

From (6)

$$c_2 = -\frac{w}{2bK}$$

From (4)

$$c_1 = \frac{w}{bK}\ell$$

$$\phi = \frac{w\ell}{Kb} x - \frac{w}{2Kb}(x^2 - z^2) + \sum_{n=1}^{\infty} a_n \cosh \frac{n\pi}{b}(\ell-x) \cos \frac{n\pi z}{b}$$

From (2) & (3)

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = \frac{w}{Kb} + \sum_{n=1}^{\infty} a_n \left(-\frac{n\pi}{a}\right) \sinh \frac{n\pi\ell}{b} \cos\frac{n\pi z}{b}$$

or

$$f(z) = \begin{cases} 0 & 0 < z < a \\ \dfrac{q}{2K(b-a)} & a < z < b \end{cases} = \frac{a_o}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi z}{b}$$

where $a_o = \dfrac{2w\ell}{Kb}$ and $A_n = -a_n \dfrac{n\pi}{b} \sinh\dfrac{n\pi\ell}{b}$

Expanding in a Fourier cosine series.

$$a_o = \frac{2}{b}\int_a^b \frac{q}{2K(b-a)}\,dz = \frac{q}{Kb}$$

Since $q = 2w\ell$, the value of a_o checks with the one already obtained

$$A_n = \frac{2}{b}\int_a^b \frac{q}{2K(b-a)} \cos\frac{n\pi z}{b}\,dz$$

$$= -\frac{q}{n\pi K(b-a)} \sin\frac{n\pi a}{b}$$

and

$$a_n = \frac{qb}{n^2\pi^2K(b-a)} \cdot \frac{\sin\frac{n\pi a}{b}}{\sinh\frac{n\pi\ell}{b}}$$

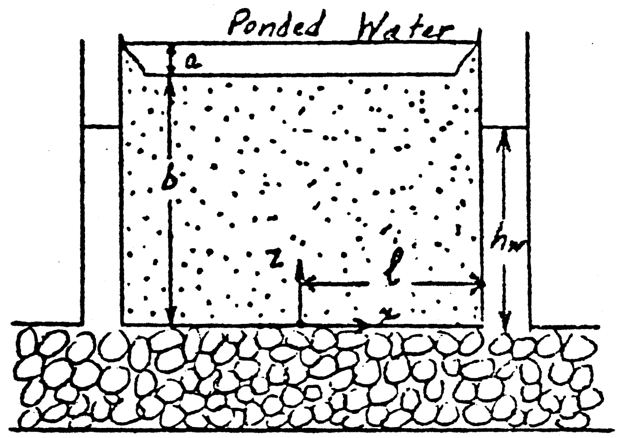
Therefore

$$\phi(x,z) = \frac{w\ell}{Kb}x - \frac{w}{2Kb}(x^2-z^2)$$

$$+ \frac{qb}{\pi^2K(b-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi a}{b} \frac{\cosh\frac{n\pi}{b}(\ell-x)}{\sinh\frac{n\pi\ell}{b}} \cos\frac{n\pi z}{b}$$

<u>Seepage into drains from a plane water table overlying a highly permeable stratum</u>

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (1)$$



$$\phi(x,0) = h_w \qquad (2)$$

$$(\phi = \frac{P}{\gamma} + z) \quad \phi(x,b) = a + b \qquad (3)$$

$$\frac{\partial \phi}{\partial x}(0,z) = 0 \qquad (4)$$

$$\left.\begin{aligned} \phi(\ell,z) &= h_w \quad & 0 < z < h_w \\ &= z \quad & h_w < z < b \end{aligned}\right\} (5)$$

Solving by separation of variables

$$\phi = c + c_1 z + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi z}{b}$$

From (2) $\quad c = h_w$

From (3) $\quad c_1 = \frac{a+b-h_w}{b}$

From (5)

$$\phi(\ell,z) = \begin{cases} h_w & 0 < z < h_w \\ z & h_w < z < b \end{cases} = h_w + \frac{a+b-h_w}{b} z + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{b}$$

where $\qquad a_n = A_n \cosh \frac{n\pi \ell}{b}$

By a Fourier sine series expansion in the interval $0 < z < b$

$$a_n = \frac{2(-1)^n}{n\pi} a - \frac{2b}{n^2\pi^2} \sin \frac{p\pi h_w}{b}$$

and

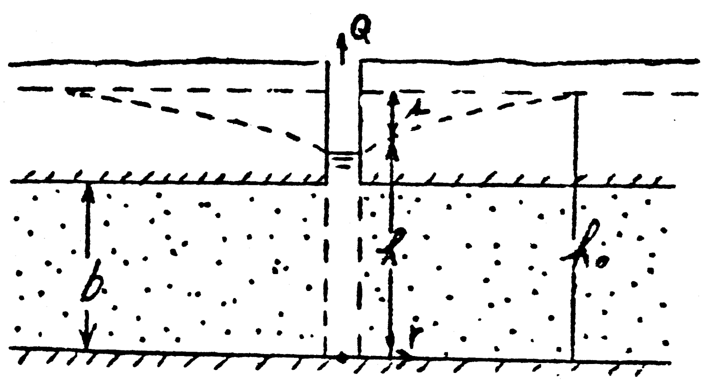
$$A_n = \frac{1}{\dfrac{1}{\cosh \frac{n\pi\ell}{b}}} \cdot a_n$$

Therefore

$$\phi(x,z) = h_w + \frac{a+b-h_w}{b} z + 2 \sum_{n=1}^{\infty} \left[\frac{a(-1)^n}{n\pi} - \frac{b}{n^2\pi^2} \sin \frac{n\pi h_w}{b}\right] \frac{\cosh \frac{n\pi x}{b}}{\cosh(n\pi\ell/b)} \sin \frac{n\pi z}{b}$$

<u>Flow toward a well in a confined aquifer</u>

Well of constant discharge.



$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0 \qquad (1)$$

$$h(r_e) = h_o \qquad (2)$$

$$\lim_{r \to 0} \; r \frac{\partial h}{\partial r} = \frac{Q}{2\pi Kb} = \frac{Q}{2\pi T} \qquad (3)$$

From (1) $$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h}{\partial r}\right) = 0$$

$$r \frac{\partial h}{\partial r} = c$$

From (3) $$c = \frac{Q}{2\pi T}$$

$$\frac{\partial h}{\partial r} = \frac{Q}{2\pi T}\frac{1}{r}$$

$$\partial h = \frac{Q}{2\pi T}\frac{\partial r}{r}$$

$$\partial h = \frac{Q}{2\pi T}\ln r + c_1$$

From (2)

$$h_o = \frac{Q}{2\pi T}\ln r_e + c_1$$

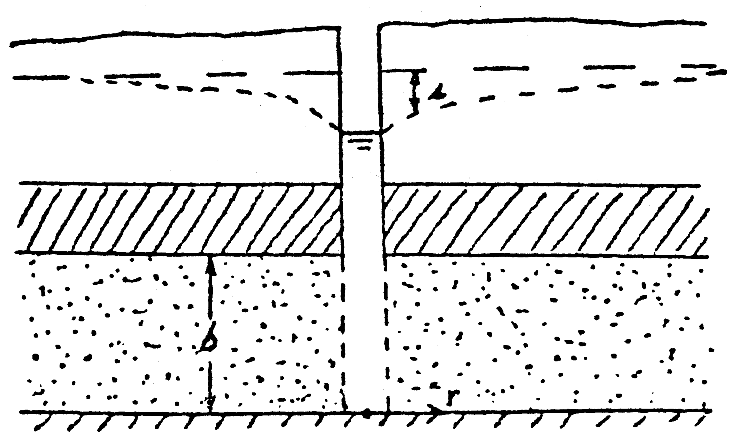
$$c_1 = h_o - \frac{Q}{2\pi T}\ln r_e$$

$$h = h_o - \frac{Q}{2\pi T}\ln\frac{r_e}{r}$$

$$s = h_o - h = \frac{Q}{2\pi T}\ln\frac{r_e}{r}$$

Flow toward a well in a leaky infinite aquifer

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = 0 \qquad (1)$$



$$s(\infty) = 0 \qquad (2)$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \qquad (3)$$

From (1) $$s = c_1 I_o\left(\frac{r}{B}\right) + c_2 K_o\left(\frac{r}{B}\right)$$

From (2) $$c_1 = 0$$

From (3) $$\lim_{r \to 0} xK_1(x) = 1$$

$$\therefore c_2 = \frac{Q}{2\pi T}$$

$$s = \frac{Q}{2\pi T} K_o\left(\frac{r}{B}\right)$$

If $s = 0$ at a distance r_e instead of ∞, condition (2) changes to $s(r_e) = 0$

$$s = c_1 I_o\left(\frac{r}{B}\right) + c_2 K_o\left(\frac{r}{B}\right)$$

$$0 = c_1 I_o\left(\frac{r_e}{B}\right) + c_2 K_o\left(\frac{r_e}{B}\right)$$

$$c_1 = - c_2 K_o\left(\frac{r_e}{B}\right) \Big/ I_o \frac{r_e}{B}$$

$$c_2 = \frac{Q}{2\pi T}$$

Therefore

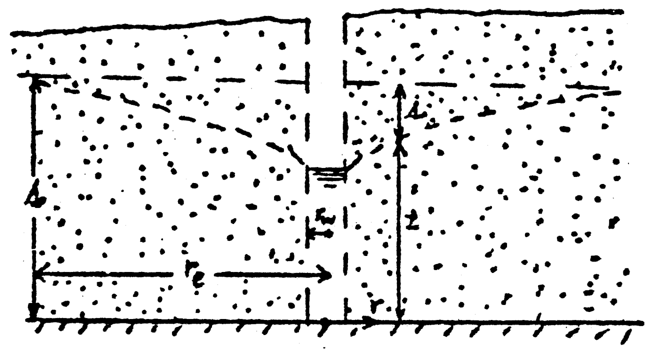
$$s = \frac{Q}{2\pi T}\left[K_o\left(\frac{r}{B}\right) - \frac{K_o(r_e/B)\ I_o(r/B)}{I_o(r_e/B)}\right]$$

Flow to a well in an unconfined aquifer

$$\frac{\partial^2 z^2}{\partial r^2} + \frac{1}{r}\frac{\partial z^2}{\partial r} = 0 \qquad (1)$$

$$z(r_e) = h_o \qquad (2)$$

$$z(r_w) = h_w \qquad (3)$$



From (1) $\quad z^2 = c \ln r + c_1$

From (2) $\quad h_o^2 = c \ln r_e + c_1$

From (3) $\quad h_w^2 = c \ln r_w + c_1$

After solving for c & c_1

$$h_o^2 - z^2 = \frac{h_o^2 - h_w^2}{\ln \frac{r_w}{r_e}} \ln \frac{r}{r_e}$$

The solution in terms of discharge is given by the Dupuit Formula

$$h_o^2 - z^2 = \frac{Q}{\pi K} \ln \frac{r_e}{r}$$

Therefore

$$Q = \frac{\pi K\left(h_o^2 - h_w^2\right)}{\ln \frac{r_e}{r_w}}$$

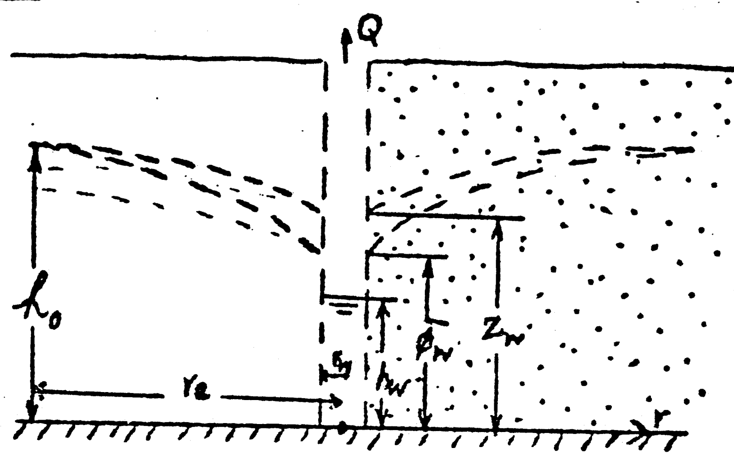
In terms of drawdown

$$h_o^2 - z^2 = (h_o - z)(h_o + z) = s(2h_o - s) = 2h_o s\left(1 - \frac{s}{2h_o}\right)$$

$$\therefore \quad s - \frac{s^2}{2h_o} = \frac{Q}{2\pi K h_o} \ln \frac{r_e}{r}$$

<u>Flow to a well in an unconfined aquifer</u>

The problem is the same as that on page 53. It will be solved this time by using the equation with $\bar{\phi}$ to show that the discharge obtained from this equation is the same as the one obtained on page 53.



(1) $$\frac{\partial^2}{\partial r^2}\left(z\bar{\phi} - \frac{z^2}{2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(z\bar{\phi} - \frac{z^2}{2}\right) = 0$$

(2) $$z(r_e) = \bar{\phi}(r_e) = h_o$$

$$Q = 2\pi r K \int_0^{z(r)} \frac{\partial \phi}{\partial r}\, dz$$

$$= 2\pi r K \left[\frac{\partial}{\partial r}\int_0^{z(r)} \phi\, dz - \phi(r,z)\frac{\partial z}{\partial r}\right]$$

(3) $$\frac{Q}{2\pi K r} = \frac{\partial}{\partial r}\left(z\bar{\phi} - \frac{z^2}{2}\right)$$

From (1) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(z\bar{\phi} - \frac{z^2}{2})\right) = 0$$

$$\frac{\partial}{\partial r}\left(z\bar{\phi} - \frac{z^2}{2}\right) = \frac{c}{r}$$

From (3) $$c = \frac{Q}{2\pi K}$$

and

$$\partial(z\bar{\phi} - \frac{z^2}{2}) = \frac{Q}{2\pi K}\frac{\partial r}{r}$$

$$z\bar{\phi} - \frac{z^2}{2} = \frac{Q}{2\pi K}\ln r + c_1$$

From (2) $$c_1 = \frac{h_o^2}{2} - \frac{Q}{2\pi K}\ln r_e$$

and $$z\bar{\phi} - \frac{z^2}{2} = \frac{h_o^2}{2} - \frac{Q}{2\pi K}\ln\frac{r_e}{r}$$

@ $r = r_w$ $\quad \phi_w = h_w \quad 0 < z < h_w$

$\quad = z \quad h_w < z < z_w$

$$z_w\bar{\phi}_w = z_w\frac{\int_0^{z_w}\phi_w dz}{z_w}$$

$$= \int_0^{h_w} h_w dz + \int_{h_w}^{z_w} z dz$$

$$= \frac{h_w^2 + z_w^2}{2}$$

$$z_w \bar{\phi}_w - \frac{z_w^2}{2} = \frac{h_o^2}{2} - \frac{Q}{2\pi K} \ln \frac{r_e}{r_w}$$

$$\frac{h_w^2 + z_w^2}{2} - \frac{z_w^2}{2} = \frac{h_o^2}{2} - \frac{Q}{2\pi K} \ln \frac{r_e}{r_w}$$

$$h_w^2 = h_o^2 - \frac{Q}{\pi K} \ln \frac{r_e}{r_w}$$

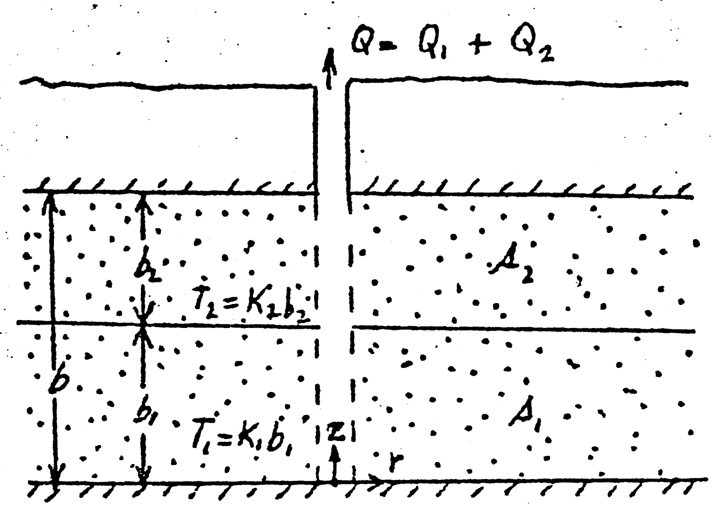
and

$$Q = \frac{\pi K \left(h_o^2 - h_w^2\right)}{\ln \frac{r_e}{r_w}}$$

which is exactly the same as the one obtained on page 53.

Flow to a well in a two layered aquifer

Lower layer:



$$\frac{\partial^2 s_1}{\partial r^2} + \frac{1}{r}\frac{\partial s_1}{\partial r} + \frac{\partial^2 s_1}{\partial z^2} = 0 \qquad (1)$$

$$\frac{\partial s_1}{\partial z}(r,0) = 0 \qquad (2)$$

$$\lim_{r \to 0} r \frac{\partial s_1}{\partial r} = -\frac{Q_1}{2\pi T_1} \qquad (3)$$

$$s_1(r_e, z) = 0 \qquad (4)$$

Solving (1) by separation of variables and choosing solutions

$$s_1 = c + c_1 \ln r + c_3 J_o(\alpha r) \cosh (\alpha z)$$

From (3) $$c_1 = - \frac{Q_1}{2\pi T_1}$$

From (4) $$c = \frac{Q_1}{2\pi T_1} \ln r_e$$

and (4) is satisfied if

$$s_1 = \frac{Q_1}{2\pi T_1} \ln \frac{r_e}{r} + \sum_{n=1}^{\infty} A_n J_o(\alpha_n r) \cosh \alpha_n z$$

where α_n are roots of $J_o(\alpha_n r_e) = 0$

Upper layer:

$$\frac{\partial^2 s_2}{\partial r^2} + \frac{1}{r}\frac{\partial s_2}{\partial r} + \frac{\partial^2 s_2}{\partial z^2} = 0 \qquad (5)$$

$$\frac{\partial s_2}{\partial z}(r,b) = 0 \qquad (6)$$

$$\lim_{r \to 0} r \frac{\partial s_2}{\partial r} = - \frac{Q_2}{2\pi T_2} \qquad (7)$$

$$s_2(r_e,z) = 0 \qquad (8)$$

All the conditions are the same as those of the lower layer, except (6), which is taken care of by changing cosh αz to cosh $\alpha(b-z)$, and

$$s_2 = \frac{Q_2}{2\pi T_2} \ln \frac{r_e}{r} + \sum_{n=1}^{\infty} B_n J_o(\alpha_n r) \cosh \alpha_n (b-z)$$

On the boundary between the two layers:

$$s_1(r,b_1) = s_2(r,b_1) \qquad (9)$$

$$\delta \frac{\partial s_1}{\partial z}(r,b_1) = \frac{\partial s_2}{\partial z}(r,b_1) \qquad (10)$$

where $\delta = \frac{K_1}{K_2}$

From (9)

$$\sum_{n=1}^{\infty} [A_n \cosh \alpha_n b_1 - B_n \cosh \alpha_n b_2] J_o(\alpha_n r) = \left[\frac{Q_2}{2\pi T_2} - \frac{Q_1}{2\pi T_1}\right] \ln \frac{r_e}{r}$$

From a Fourier-Bessel series expansion of the right side

$$A_n \cosh \alpha_n b_1 - B_n \cosh \alpha_n b_2 = \frac{2}{[\alpha_n r_e J_1(\alpha_n r_e)]^2} \left[\frac{Q_2}{2\pi T_2} - \frac{Q_1}{2\pi T_1}\right]$$

From (10)

$$\delta A_n \sinh \alpha_n b_1 + B_n \sinh \alpha_n b_2 = 0$$

After solving for A_n and B_n

$$s_1 = \frac{Q_1}{2\pi T_1} \ln \frac{r_e}{r} + 4\left(\frac{Q_2}{2\pi T_2} - \frac{Q_1}{2\pi T_1}\right) \sum_{n=1}^{\infty} D_n \sinh \alpha_n b_2 \cosh \alpha_n z J_o(\alpha_n r)$$

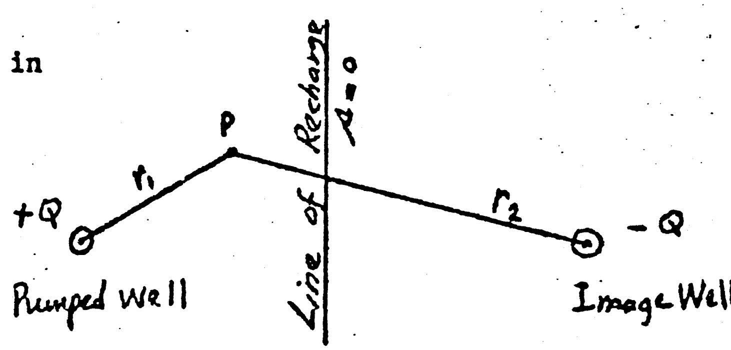
$$s_2 = \frac{Q_2}{2\pi T_2} \ln \frac{r_e}{r} + 4\delta\left(\frac{Q_1}{2\pi T_1} - \frac{Q_2}{2\pi T_2}\right) \sum_{n=1}^{\infty} D_n \sinh \alpha_n b_1 \cosh \alpha_n (b-z) J_o(\alpha_n r)$$

where $$D_n = \frac{1}{[\alpha_n r_e J_1(\alpha_n r_e)]^2 \; [(1+\delta)\sinh \alpha_n b - (1-\delta)\sinh \alpha_n (b_1 - b_2)]}$$

Flow to a well near a line of recharge

The equation for flow to a well in a confined aquifer was found to be

$$s = \frac{Q}{2\pi T} \ln \frac{r_e}{r}$$



Along the line of recharge the water level is maintained constant and therefore an equation for a well near a line of recharge should satisfy the condition $s = 0$ at any point on the line of recharge.

The method of images is used and zero drawdown along the line of recharge is obtained by a recharging image well located on the other side of the recharge line at a distance equal to that between the real well and the line of recharge.

Assuming that r_e is very large compared to the distance between the real and image well, the resulting drawdown at any point will be given by:

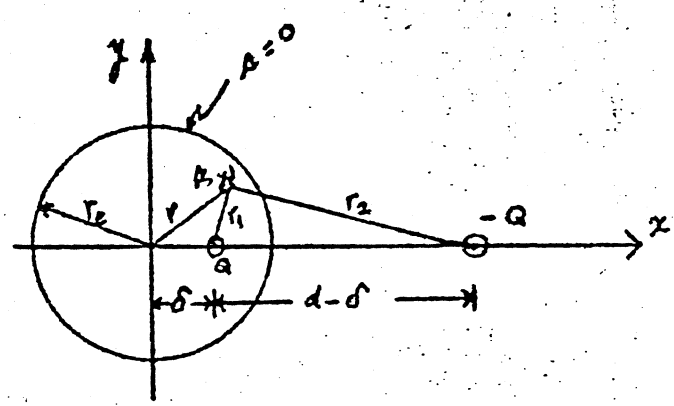
$$s = \frac{Q}{2\pi T} \ln \frac{r_e}{r_1} - \frac{Q}{2\pi T} \ln \frac{r_e}{r_2}$$

$$= \frac{Q}{2\pi T} \ln \frac{r_2}{r_1}$$

Along the line of recharge $r_2 = r_1$ and $s = 0$

Flow to an eccentric well in a circular island

The method of images is used again to satisfy the condition of zero drawdown on the outside boundary of the island. A recharging image well is placed at a distance d from the center of the island. The resulting drawdown equation should have the form:



$$s = \frac{Q}{2\pi T} \ln c \frac{r_2}{r_1}$$

where c is a constant that makes $s = 0$ on the boundary of the island. The problem is to determine this constant c, and the distance d, in terms of known quantities.

On the boundary:

$$s = 0 \qquad \therefore \qquad \ln c \frac{r_2}{r_1} = 0$$

and

$$\frac{c r_2}{r_1} = 1$$

$$\frac{r_2^{\,2}}{r_1^{\,2}} = \frac{1}{c^2}$$

$$\frac{(x-d)^2 + y^2}{(x-\delta)^2 + y^2} = \frac{1}{c^2}$$

$$c^2(x^2 - 2xd + d^2 + y^2) = x^2 - 2x\delta + \delta^2 + y^2$$

$$c^2x^2 - 2c^2xd + c^2d^2 + c^2y^2 = x^2 - 2x\delta + \delta^2 + y^2$$

$$(1-c^2)x^2 - 2x(\delta-c^2d) + (1-c^2)y^2 = c^2d^2 - \delta^2$$

$$x^2 - 2x\,\frac{(\delta-c^2d)}{1-c^2} + y^2 = \frac{c^2d^2-\delta^2}{1-c^2}$$

By completing the square

$$\left(x - \frac{\delta-c^2d}{1-c^2}\right)^2 + y^2 = \frac{c^2d^2-\delta^2}{1-c^2} + \left(\frac{\delta-c^2d}{1-c^2}\right)^2$$

$$= \frac{c^2(d-\delta)^2}{(1-c^2)^2}$$

Which is a family of circles with center at:

$$\left(\frac{\delta - c^2 d}{1 - c^2} , 0 \right)$$

and radius $$r = \frac{c(d - \delta)}{1 - c^2}$$

But we know that the center of the island is at (0,0) and its radius is r_e. Therefore

$$\frac{\delta - c^2 d}{1 - c^2} = 0$$

and

$$\frac{c(d - \delta)}{1 - c^2} = r_e$$

Solving for c and d we obtain

$$c = \frac{\delta}{r_e} \qquad\qquad d = \frac{r_e^2}{\delta}$$

Therefore the image well should be placed at a distance r_e^2/δ from the center of the island, and the drawdown at any point on the island will be given by

$$s = \frac{Q}{2\pi T} \ln \frac{\delta}{r_e} \frac{r_2}{r_1}$$

or,

$$s = \frac{Q}{4\pi T} \ln \frac{\delta^2}{r_e^2} \frac{\left(x - \frac{r_e^2}{\delta}\right)^2 + y^2}{(x-\delta)^2 + y^2}$$

As $\delta \to 0$, i.e. the well becomes concentric:

$$s = \frac{Q}{4\pi T} \ln_{\delta \to 0} \frac{1}{r_e^2} \frac{(\delta x - r_e^2)^2 + \delta^2 y^2}{(x-\delta)^2 + y^2}$$

$$= \frac{Q}{4\pi T} \ln \frac{r_e^4}{r_e^2\, r^2}$$

$$= \frac{Q}{2\pi T} \ln \frac{r_e}{r}$$

The ratio of the discharges for the same drawdown, in an eccentric and a concentric well will be as follows:

For a concentric well

$$s_o = \frac{Q_o}{4\pi T} \ln \frac{r_e^2}{r^2}$$

For an eccentric well

$$s_e = \frac{Q_e}{4\pi T} \ln \frac{(\delta x - r_e^2)^2 + \delta^2 y^2}{[(x-\delta)^2 + y^2] r_e^2}$$

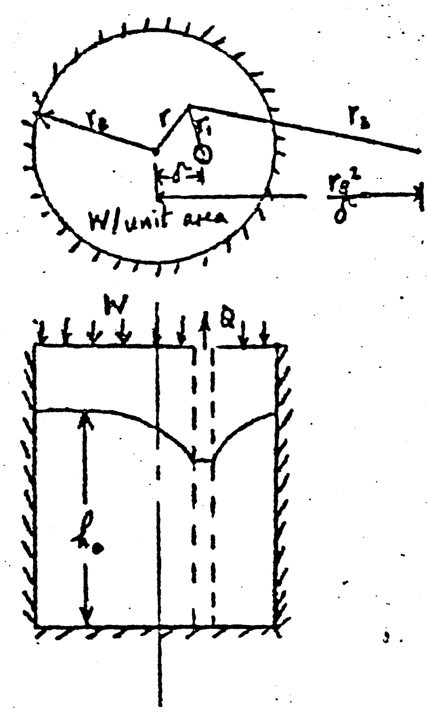
If $s_o = s_e$

$$Q_o/Q_e = \left\{ \ln(r_e^2/r^2) + \ln(r^2/r_e^4) \left[\left((\delta_x - r_e^2)^2 + \delta^2 y^2 \right) \Big/ \left((x-\delta)^2 + y^2 \right) \right] \right\} \Big/ \ln(r_e^2/r^2)$$

$$= 1 + \left\{ \ln r^2 \left[\left((x\delta/r_e^2) - 1 \right)^2 + y^2 (\delta/r_e^2)^2 \right] \Big/ \left((x-\delta)^2 + y^2 \right) \right\} \Big/ \ln(r_e^2/r^2)$$

<u>Flow to an eccentric well in a closed circular aquifer in balance with rainfall</u>

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} = \beta \qquad (1)$$



$$\frac{\partial h}{\partial r}(r_e, \theta) = 0 \qquad (2)$$

$$\lim_{r_1 \to 0} r_1 \frac{\partial h}{\partial r_1} = \frac{Q}{2\pi T} \qquad (3)$$

where $\beta = -\dfrac{W}{Kh_o}$

One of the particular solutions is

$$h = \frac{Q}{2\pi T} \ln r_1$$

Substituting $h = \phi - \frac{\beta}{4} r^2$ on (1) we obtain

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Let $\phi = R\theta$ and solve by separation of variables particular solutions

$\phi = r^\alpha \sin\alpha\theta,\ r^\alpha \cos\alpha\theta,\ \dfrac{\sin\alpha\theta}{r^\alpha},\ \dfrac{\cos\alpha\theta}{r^\alpha}$

Particular solutions for h:

$$h = \frac{Q}{2\pi T}\ln r_1,\ r^\alpha \sin\alpha\theta - \frac{\beta r^2}{4},\ r^\alpha\cos\alpha\theta - \frac{\beta r^2}{4},\ \frac{\sin\alpha\theta}{r^\alpha} - \frac{\beta r^2}{4},$$

$$\frac{\cos\alpha\theta}{r^\alpha} - \frac{\beta r^2}{4}$$

At $r = 0$, $h \neq 0$ $\therefore$ we drop $1/r^\alpha$ terms. The sine terms = 0 for $\theta = 0$ & π $\therefore$ eliminate. As θ is periodic we choose $\alpha = n$ $(n = 1, 2, 3, \ldots)$

$$h = \frac{Q}{2\pi T}\ln r_1 - \frac{\beta r^2}{4} + \sum_{n=1}^{\infty}\frac{Q}{2\pi T}a_n r^n \cos n\theta$$

$$= \frac{Q}{2\pi T}\left[\ln r_1 - \frac{\beta r^2}{(4Q/2\pi T)} + \sum_{n=1}^{\infty} a_n r^n \cos n\theta\right]$$

Use $c = \dfrac{Q}{2\pi T}$

From (2)

$$\left.\frac{\partial h}{\partial r}\right|_{r=r_e} = c\left\{\frac{\partial}{\partial r}\left[\ln\sqrt{r^2+\delta^2-2r\delta\cos\theta} - \frac{2\beta r}{4c}\right.\right.$$

$$\left.\left. + \sum_{n=1}^{\infty} a_n n r^{n-1}\cos n\theta\right]\right\}_{r=r_e} = 0$$

$$= c\left[\frac{r_e - \delta\cos\theta}{r_e^2 + \delta^2 - 2r_e\delta\cos\theta} - \frac{\beta r_e}{2c}\right.$$

$$\left. + \sum_{n=1}^{\infty} a_n n r_e^{n-1}\cos n\theta\right] = 0$$

consider

$$\frac{r_e - \delta\cos\theta}{r_e^2+\delta^2-2r_e\delta\cos\theta}$$

From complex number theory

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for} \quad |z| < 1$$

$$z = x + iy = Re^{i\theta} = R(\cos\theta + i\sin\theta)$$

$$\frac{1}{1-R(\cos\theta + i\sin\theta)} = \frac{1}{(1-R\cos\theta) - i R\sin\theta} = \sum_{n=0}^{\infty} R^n e^{in\theta}$$

$$\frac{(1-R\cos\theta)+i R\sin\theta}{(1-R\cos\theta)^2+R^2\sin^2\theta} = \sum_{n=0}^{\infty} R^n (\cos n\theta + i\sin n\theta)$$

$$\frac{1-R\cos\theta}{1+R^2-2R\cos\theta} + i\,\frac{R\sin\theta}{1+R^2-2R\cos\theta} =$$

$$\sum_{n=0}^{\infty} R^n \cos n\theta + i \sum_{n=0}^{\infty} R^n \sin n\theta$$

Therefore,

$$\left.\begin{aligned} \frac{1-R\cos\theta}{1+R^2-2R\cos\theta} &= \sum_{n=0}^{\infty} R^n \cos n\theta \\ \frac{R\sin\theta}{1+R^2-2R\cos\theta} &= \sum_{n=0}^{\infty} R^n \sin\theta \end{aligned}\right\} \text{for } R < 1$$

and

$$\frac{r_e - \delta \cos\theta}{r_e^2+\delta^2-2r_e\,\delta\cos\theta} = \frac{1}{r_e}\left(\frac{1-(\delta/r_e)\cos\theta}{1+(\delta/r_e)^2-2(\delta/r_e)\cos\theta}\right) = \frac{1}{r_e}\sum_{n=0}^{\infty}\left(\frac{\delta}{r_e}\right)^n \cos n\theta$$

$$\left.\frac{\partial h}{\partial r}\right|_{r=r_e} = c\left[\frac{1}{r_e}\sum_{n=0}^{\infty}\left(\frac{\delta}{r_e}\right)^n \cos n\theta - \frac{\beta r_e}{2c} + \sum_{n=1}^{\infty} a_n n r_e^{n-1}\cos n\theta\right] = 0$$

or

$$\frac{1}{r_e} + \frac{1}{r_e}\sum_{n=1}^{\infty}\left(\frac{\delta}{r_e}\right)^n \cos n\theta - \frac{\beta r_e}{2c} + \sum_{n=1}^{\infty} a_n n r_e^{n-1}\cos n\theta = 0$$

Therefore $\frac{\beta r_e}{2c} = \frac{1}{r_e}$ and $n a_n r_e^{n-1} = -\frac{1}{r_e}\frac{\delta^2}{r_e^n}$

$$c = \frac{\beta}{2} r_e^2 = \frac{w r_e^2}{2T} \times \frac{\pi}{\pi} = \frac{Q}{2\pi T}$$

$$a_n = -\frac{1}{n}\left(\frac{\delta}{r_e^2}\right)^n$$

and

$$h = c\ln r_1 - \frac{\beta r^2}{4} - c\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\delta r}{r_e^2}\right)^n \cos n\theta$$

But $\ln r_1$ has a dimension; to make it dimensionless we have to introduce another constant using another condition:

$$@ \quad r_1 = r_w \qquad h = h_w$$

Before applying this condition let's simplify the solution and add a constant A. From complex theory:

$$\int \frac{1}{1-z}\, dz = \int \sum_{n=0}^{\infty} z^n dz$$

$$\ln(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} = -\left[\sum_{n=1}^{\infty} \frac{R^n}{n} (\cos n\theta + i \sin n\theta)\right]$$

where $\qquad z = R(\cos\theta - i\sin\theta)$

but $|1-z| = R_1 = \sqrt{1+R^2-2R\cos\theta}$

and $\qquad \ln(1-z) = \ln R_1 e^{i\theta} = \ln R_1 + i\theta$

where $\qquad \phi = \dfrac{R\sin\theta}{1-R\cos\theta}$

$$\ln R_1 = \frac{1}{2}\ln(1+R^2-2R\cos\theta) = -\sum_{n=1}^{\infty} \frac{R^n}{n} \cos n\theta$$

$$\phi = -\sum_{n=1}^{\infty} \frac{R^n}{n} \sin n\theta$$

Therefore

$$h = c \ln r_1 - \frac{\beta r^2}{4} + \frac{c}{2} \ln \left[1 + \left(\frac{\delta r}{r_e^2} \right)^2 - \frac{2\delta r}{r_e^2} \cos\theta \right] + A$$

$$= c \ln r_1 + A - \frac{\beta r^2}{4} + \frac{c}{2} \ln \frac{\delta^2}{r_e^4} \underbrace{\left[\left(\frac{r_e^2}{\delta} \right)^2 + r^2 + \frac{2r_e^2}{\delta} r \cos\theta \right]}_{r_2^2}$$

$$h = c \ln \frac{r_1 r_2 \delta}{r_e^2} - \frac{\beta r^2}{4} + A$$

$$@\ r_1 = r_w;\ h = h_w \quad , \quad r_2 = \frac{r_e^2 - \delta^2}{\delta} , \qquad r = \delta \pm r_w$$

$$h_w = c \ln \frac{r_w \delta}{r_e^2} \cdot \left(\frac{r_e^2 - \delta^2}{\delta} \right) - \frac{\beta(\delta \pm r_w)}{4} + A$$

After solving for A, and replacing C and B.

$$h - h_w = \frac{Q}{2\pi T} \ln \frac{r_1 r_2 \delta}{r_w (r_e^2 - \delta^2)} - \frac{W}{4T} [r^2 - (\delta \pm r_w)^2]$$

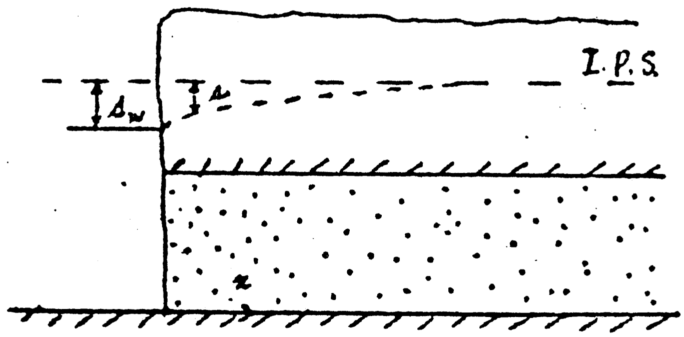
where $$r_1^2 = r^2 + \delta^2 - 2\delta r \cos\theta$$

$$r_2^2 = r^2 + (r_e^2/\delta)^2 - 2r(r_e^2/\delta) \cos\theta$$

NONSTEADY FLOW PROBLEMS

Flow to a line of recharge

To find the rate of flow from the aquifer to the lake when the water level in the lake has dropped suddenly.



$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (1)$$

$$\nu = \frac{K}{S_s}$$

$$s(0,t) = s_w \qquad (2)$$

$$s(\infty,t) = 0 \qquad (3)$$

$$s(x,0) = 0 \qquad (4)$$

$$s(x,\infty) = s_w \qquad (5)$$

By the method of separation of variables (1) will give:

$$s = c_1 \; e^{-\alpha^2 \nu t} \begin{cases} \sin \alpha x \\ \cos \alpha x \end{cases}$$

From (2), and including all values of α

$$s = s_w + \int_0^\infty A(\alpha) e^{-\alpha^2 \nu t} \sin \alpha x \, d\alpha$$

we see that (5) is also satisfied. From (4)

$$0 = s_w + \int_0^\infty A(\alpha) \sin \alpha x \, d\alpha$$

From a table of integrals

$$\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha = \frac{\pi}{2}$$

$$\frac{2s_w}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha = s_w$$

$$0 = \frac{2s_w}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha + \int_0^\infty A(\alpha)\sin \alpha x\, d\alpha$$

$$0 = \int_0^\infty \left[\frac{2}{\pi}\frac{s_w}{\alpha} + A(\alpha)\right]\sin \alpha x\, d\alpha$$

$$A(\alpha) = -\frac{2}{\pi}\frac{s_w}{\alpha}$$

$$s = s_w - \frac{2}{\pi}s_w\int_0^\infty \frac{e^{-\alpha^2 \nu t}}{\alpha}\sin \alpha x\, d\alpha$$

From a table of integrals

$$\left[\int_0^\infty e^{-t\mu^2}\cos\mu b\, d\mu\right] = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-\frac{b^2}{4t}}$$

$$\int_0^b \left[\int_0^\infty e^{-t\mu^2}\cos\mu b\, d\mu\right] db = \frac{1}{2}\int_0^b \sqrt{\frac{\pi}{t}}\, e^{-\frac{b^2}{4t}}\, db$$

$$\int_0^\infty \frac{e^{-t\mu^2}}{\mu}\sin\mu b\, d\mu = \frac{1}{2}\sqrt{\frac{\pi}{t}}\int_0^b e^{-\frac{b^2}{4t}}\, db$$

$$\text{Let } \frac{b^2}{4t} = \beta^2$$

then $b = 2\sqrt{t}\beta$, and $db = 2\sqrt{t}d\beta$

Limits: @ $b = 0$ $\beta = 0$

@ $b = b$ $\beta = \frac{b}{2\sqrt{t}}$

$$\int_0^\infty \frac{e^{-t\mu^2}}{\mu} \sin\mu b d\mu = \sqrt{\pi} \int_0^{\frac{b}{\sqrt{4t}}} e^{-\beta^2} d\beta$$

$$= \frac{\pi}{2}\left[\frac{2}{\sqrt{\pi}} \int_0^{\frac{b}{\sqrt{4t}}} e^{-\beta^2} d\beta\right]$$

$$= \frac{\pi}{2} \operatorname{erf}\left(\frac{b}{\sqrt{4t}}\right)$$

Therefore,

$$\int_0^\infty \frac{e^{-\alpha^2 \nu t}}{\alpha} \sin\alpha x d\alpha = \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

and $$s = s_w - \frac{2}{\pi} s_w \left[\frac{\pi}{2} \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)\right]$$

we see that (3) is also satisfied.

$$s = s_w \left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)\right]$$

$$s = s_w \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

The rate of flow

$$q_x = -T \frac{\partial s}{\partial x}$$

$$\frac{\partial s}{\partial x} = \bar{s}_w \frac{\partial}{\partial x}\left[\operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)\right]$$

$$= s_w \frac{\partial}{\partial x}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} e^{-\beta^2}\, d\beta\right]$$

$$= \frac{2s_w}{\sqrt{\pi}}\left[-e^{-\frac{x^2}{4\nu t}} \cdot \frac{1}{\sqrt{4\nu t}}\right]$$

$$= -\frac{s_w}{\sqrt{\nu\pi t}}\, e^{-\frac{x^2}{4\nu t}}$$

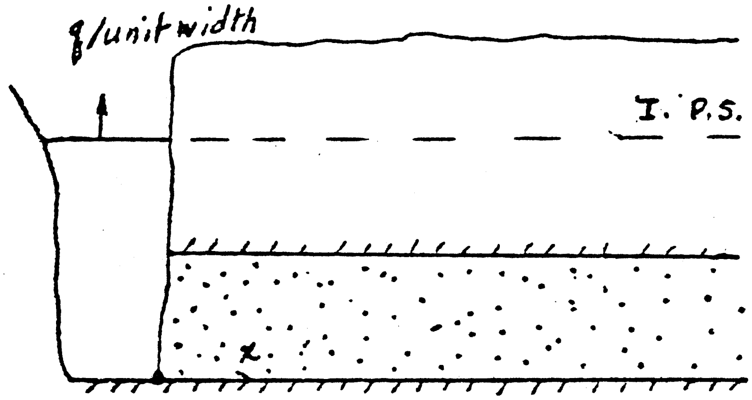
$$q_x = \frac{Ts_w}{\sqrt{\nu\pi t}}\, e^{-\frac{x^2}{4\nu t}}$$

@ $x = 0$

$$q = \frac{Ts_w}{\sqrt{\nu\pi t}}$$

Flow to a line source of constant discharge

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (1)$$



$$s(x,0) = 0 \qquad (2)$$

$$s(\infty,t) = 0 \qquad (3)$$

$$\frac{\partial s}{\partial x}(0,t) = -\frac{q}{T} \qquad (4)$$

Solving (1) by separation of variables

$$s = c, \quad c_1 x \quad , \quad e^{-\nu\alpha^2 t} \begin{cases} \sin \alpha x \\ \cos \alpha x \end{cases}$$

sin αx is eliminated because of (4)

$$s = c + c_1 x + c_1 \int_0^\infty A(\alpha) e^{-\nu\alpha^2 t} \cos \alpha x d\alpha$$

Note: Usually if we have constant drawdown we use the sine term, if we have constant discharge we use the cosine term.

From (4)

$$c_1 = -\frac{q}{T}$$

$$s = c - \frac{q}{T} x - \frac{q}{T} \int_0^\infty A(\alpha)\, e^{-\nu\alpha^2 t} \cos \alpha x d\alpha$$

From (3)

$0 = -\infty -$ (a finite value which can be incorporated in $-\infty$)

$\therefore \quad c = \infty$

$$s = \frac{q}{1}\left[\infty - x - \int_0^\infty A(\alpha)e^{-\nu\alpha^2 t}\cos\alpha x d\alpha\right]$$

From (2)

$$0 = \infty - x - \int_0^\infty A(\alpha)\cos\alpha x d\alpha$$

We have to represent $(\infty - x)$ in the form of an infinite integral, containing cosine terms.

We know $\quad \int_0^\infty \frac{\sin\alpha x}{\alpha} d\alpha = \frac{\pi}{2}$

Integrate w.r. to x

$$\int_0^x\left[\int_0^\infty \frac{\sin\alpha x}{\alpha} d\alpha\right] dx = \int_0^x \frac{\pi}{2} dx$$

$$\int_0^\infty \frac{d\alpha}{\alpha}\int_0^x \sin\alpha x dx = \frac{\pi}{2}\ x$$

$$\int_0^\infty \frac{d\alpha}{\alpha}\left(-\frac{\cos\alpha x}{\alpha}\right)_0^x = \int_0^\infty \frac{d\alpha}{\alpha^2}(1-\cos\alpha x) = \frac{\pi}{2}x$$

$$\int_0^{\infty} \frac{d\alpha}{\alpha^2} - \int_0^{\infty} \frac{\cos \alpha x d\alpha}{\alpha^2} = \frac{\pi}{2} x$$

$$\left[-\frac{1}{\alpha}\right]_0^{\infty} - \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2} d\alpha = \frac{\pi}{2} x$$

$$= -\int_0^{\infty} \frac{\cos\alpha x}{\alpha^2} d\alpha = \frac{\pi}{2} x$$

or

$$\frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2} d\alpha = x$$

$$\therefore \quad \frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2} d\alpha - \int_0^{\infty} A(\alpha) \cos \alpha x d\alpha = 0$$

and $$A(\alpha) = \frac{2}{\pi\alpha^2}$$

$$s = \frac{q}{T}\left[\frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2} d\alpha - \frac{2}{\pi} \int_0^{\infty} \frac{e^{-\nu\alpha^2 t}}{\alpha^2} \cos \alpha x d\alpha\right]$$

$$s = \frac{q}{T} \frac{2}{\pi} \int_0^{\infty} \frac{1-e^{-\nu\alpha^2 t}}{\alpha^2} \cos \alpha x d\alpha$$

We know that

$$\int_0^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x d\alpha = \frac{\sqrt{\pi}}{2} \frac{e^{-\frac{x^2}{4\nu t}}}{\sqrt{\nu t}}$$

Let $\nu t = y$ and integrate w.r. to y from 0 to νt. We obtain

$$\int_0^\infty \frac{1-e^{-\nu\alpha^2 t}}{\alpha^2} \cos \alpha x d\alpha = \frac{\sqrt{\pi}}{2} \int_0^{\nu t} \frac{e^{-\frac{x^2}{4y}}}{\sqrt{y}} dy$$

$$s = \frac{q}{T} \frac{1}{\sqrt{\pi}} \int_0^{\nu t} \frac{e^{-\frac{x^2}{4y}}}{\sqrt{y}} dy$$

The substitution $\frac{x^2}{4y} = \beta^2$, leads to:

$$s = \frac{q}{T} \frac{x}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} \frac{e^{-\beta^2}}{\beta^2} d\beta$$

Integrating by parts $\left(u = e^{-\beta^2},\ dv = \frac{d\beta}{\beta^2}\right)$

$$s = \frac{q}{T} \frac{x}{\sqrt{\pi}} \left\{ \left[-\frac{e^{-\beta^2}}{\beta} \right]_{\frac{x}{\sqrt{4\nu t}}}^{\infty} - 2 \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} e^{-\beta^2} d\beta \right\}$$

$$s = \frac{q}{T} \frac{x}{\sqrt{\pi}} \left[\frac{e^{-\frac{x^2}{4\nu t}}}{\frac{x}{\sqrt{4\nu t}}} - \sqrt{\pi}\ \text{erfc} \left(\frac{x}{\sqrt{4\nu t}} \right) \right]$$

Let $\frac{x^2}{4\nu t} = u_x$

$$s = \frac{q}{T}\frac{x}{\sqrt{\pi}}\left[\frac{e^{-u_x}}{\sqrt{u_x}} - \sqrt{\pi}\ \mathrm{erfc}\ (\sqrt{u_x})\right]$$

The problem can also be solved in a simpler way, by making it similar to the previously solved problem of constant drawdown.

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial s}{\partial x}\right) = \frac{1}{\nu}\frac{\partial}{\partial t}(s)$$

At any point: $q_x = -T\frac{\partial s}{\partial x}$

$\therefore$

$$\frac{\partial}{\partial x}\left(-T\frac{\partial s}{\partial x}\right) = \frac{1}{\nu}\frac{\partial}{\partial t}(-Ts)$$

$$\frac{\partial q_x}{\partial x} = \frac{1}{\nu}\frac{\partial}{\partial t}(-Ts)$$

Taking the derivative w.r. to x

$$\frac{\partial^2 q_x}{\partial x^2} = \frac{1}{\nu}\frac{\partial}{\partial t}\left(-T\frac{\partial s}{\partial x}\right)$$

$$\frac{\partial^2 q_x}{\partial x^2} = \frac{1}{\nu}\frac{\partial q_x}{\partial t} \qquad (1)$$

$$q_x(0,t) = q \qquad (2)$$

$$q_x(\infty,t) = 0 \qquad (3)$$

$$q_x(x,0) = 0 \qquad (4)$$

From similarity to the constant drawdown problem (see page)

$$q_x = q \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

$$-\frac{\partial s}{\partial x} = \frac{q}{T} \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

Integrating both sides w.r. to x between the limits x and infinity

$$-\int_x^\infty \frac{\partial s}{\partial x}\,dx = \frac{q}{T}\int_x^\infty \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)dx$$

$$s = \frac{q}{T}\int_x^\infty \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)dx$$

Let $\dfrac{x}{\sqrt{4\nu t}} = \beta$

$$s = \frac{q}{T} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} \operatorname{erfc}(\beta)\, d\beta \sqrt{4\nu t}$$

$$= \frac{q}{T} \sqrt{4\nu t}\ \mathrm{i\,erfc}\ \frac{x}{\sqrt{4\nu t}}$$

But $\mathrm{i\,erfc}\ (\beta) = \frac{1}{\sqrt{\pi}} e^{-\beta^2} - \beta \operatorname{erfc} \beta$

$$\mathrm{i\,erfc}\ \frac{x}{\sqrt{4\nu t}} = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4\nu t}} - \frac{x}{\sqrt{4\nu t}} \operatorname{erfc} \frac{x}{\sqrt{4\nu t}}$$

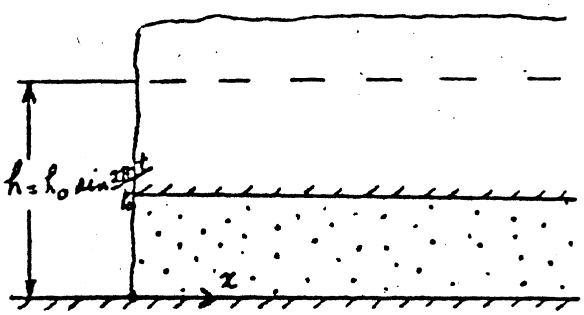
and after some manipulation

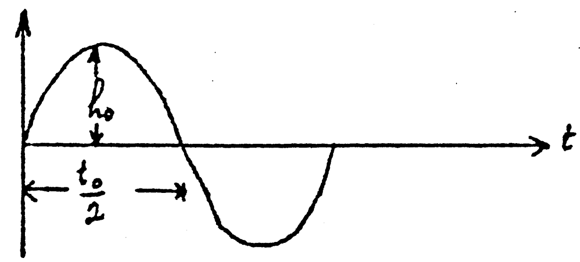
$$s = \frac{q}{T} \frac{x}{\sqrt{\pi}} \quad \frac{e^{-\frac{x^2}{4\nu t}}}{\frac{x}{\sqrt{4\nu t}}} - \sqrt{\pi} \operatorname{erfc} \frac{x}{\sqrt{4\nu t}}$$

Let $\qquad \frac{x^2}{4\nu t} = u_x$

$$s = \frac{q}{T} \frac{x}{\sqrt{\pi}} \left[\frac{e^{-u_x}}{\sqrt{u_x}} - \sqrt{\pi} \operatorname{erfc} [\sqrt{u_x}) \right]$$

Fluctuation of water levels in response to tidal fluctuations.





$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(0,t) = h_o \sin \frac{2\pi t}{t_o} \qquad (2)$$

$$h(\infty,t) = 0 \qquad (3)$$

By separation of variables

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{\nu T}\frac{\partial T}{\partial t}$$

Cond. (2) suggests that we need a sine function of t. To accomplish this we equate the above equation to an imaginary constant.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{\nu\tau}\frac{\partial \tau}{\partial t} = i\alpha$$

$$X = e^{\pm x\sqrt{i\alpha}}$$

$$\tau = e^{i\nu\alpha t}$$

$$h = e^{\pm x\sqrt{i\alpha} + i\nu\alpha t} \quad , \quad e^{\pm x\sqrt{-i\alpha} - i\nu\alpha t}$$

From complex variables

$$z = re^{i\theta}$$

$$i = e^{i\,\pi/2}$$

$$\sqrt{i} = e^{i\,\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$$

$$\sqrt{-i} = e^{-i\,\pi/4} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1-i)$$

$$h = e^{\pm x\sqrt{\frac{\alpha}{2}}(1+i)+i\nu at} \quad , \quad e^{\pm x\sqrt{\frac{\alpha}{2}}(1-i)-i\nu at}$$

$$= e^{\pm x\sqrt{\frac{\alpha}{2}}} e^{\pm ix\sqrt{\frac{\alpha}{2}}+i\nu at} \quad , \quad e^{\pm ix\sqrt{\frac{\alpha}{2}}} e^{\pm x\sqrt{\frac{\alpha}{2}}-i\nu at}$$

$$= e^{\pm x\sqrt{\frac{\alpha}{2}}}\left[c_1 e^{i(\pm x\sqrt{\frac{\alpha}{2}}+\nu at)} + c_2\, e^{-i(\pm x\sqrt{\frac{\alpha}{2}}+\nu at)}\right]$$

$$= e^{x\sqrt{\frac{\alpha}{2}}}\begin{cases}\sin(\nu at + x\sqrt{\frac{\alpha}{2}}) \\ \cos(\nu at + x\sqrt{\frac{\alpha}{2}})\end{cases} \quad , \quad e^{-x\sqrt{\frac{\alpha}{2}}}\begin{cases}\sin(\nu at - x\sqrt{\frac{\alpha}{2}}) \\ \cos(\nu at - x\sqrt{\frac{\alpha}{2}})\end{cases}$$

By the help of the boundary conditions we choose:

$$h = ce^{-x\sqrt{\frac{\alpha}{2}}} \sin(\nu at - x\sqrt{\frac{\alpha}{2}})$$

we see that cond (3) is satisfied. From (2)

$$h_o \sin\frac{2\pi t}{t_o} = c \sin \nu at$$

$$c = h_o \qquad \alpha = \frac{2\pi}{\nu t_o}$$

and

$$h = h_o e^{-x\sqrt{\frac{\pi}{\nu t_o}}} \sin\left(\frac{2\pi t}{t_o} - x\sqrt{\frac{\pi}{\nu t_o}}\right)$$

Important: This equation gives the water levels after the fluctuations have reached a steady state.

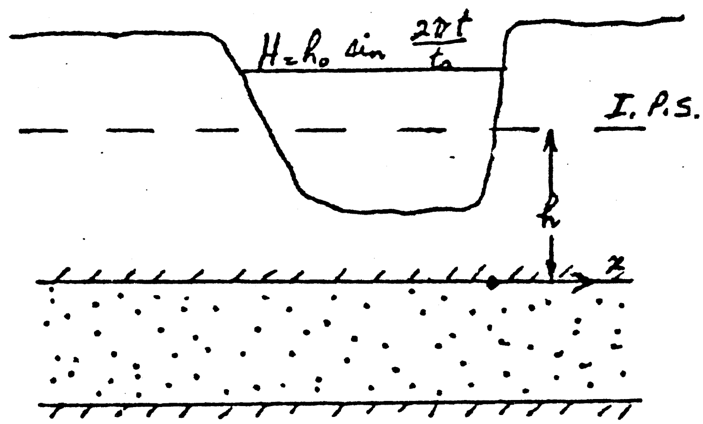
The amplitude dies exponentially with increasing x. The time lag or phase difference is given by:

$$\frac{2\pi t}{t_o} - x\sqrt{\frac{\pi}{\nu t_o}} = 0$$

$$t = \frac{x}{2}\sqrt{\frac{t_o}{\pi\nu}}$$

A special case is when the fluctuating surface water body is not in contact with the lower aquifer.

The relation between changes in H and h is given by



$$\frac{dh}{dH} = \frac{\alpha}{\alpha+\theta\beta}$$

Therefore the amplitude of the fluctuation in the surface water body will be transmitted to the lower aquifer diminished by that factor. In other words condition (2) will change to:

$$h(0,t) = \frac{\alpha}{\alpha+\theta\beta} h_o \sin \frac{2\pi t}{t_o}$$

and the equation for this case will be:

$$h = \frac{\alpha}{\alpha+\theta\beta} h_o e^{-x\sqrt{\frac{\pi}{\nu t_o}}} \sin\left(\frac{2\pi t}{t_o} - x\sqrt{\frac{\pi}{\nu t_o}}\right)$$

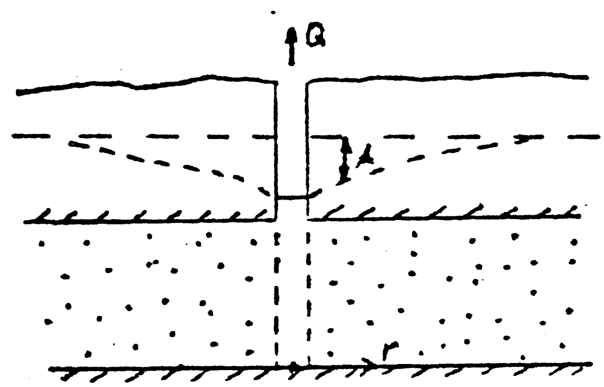
Flow to a well in an infinite confined aquifer

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (1)$$

$$s(\infty,t) = 0 \qquad (2)$$

$$s(r,0) = 0 \qquad (3)$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \qquad (4)$$



By separation of variables

$$s = e^{-\nu\alpha^2 t}\begin{cases} J_0(\alpha r) \\ Y_0(\alpha r)\end{cases}, \quad \ln r, \quad c$$

Condition (4) suggests elimination of $Y_0(\alpha r)$

$$s = c + c_1 \ln r + c_1 \int_0^\infty A(\alpha) J_0(\alpha r) e^{-\alpha^2 \nu t} \, d\alpha$$

From (4)

$$c_1 = -\frac{Q}{2\pi T}$$

From (2)

$$c = -\infty c_1$$

$$s = \frac{Q}{2\pi T}\left[\infty - \ln r - \int_0^\infty A(\alpha) J_0(\alpha r) e^{-\alpha^2 \nu t} \, d\alpha\right]$$

From (3)

$$0 = \frac{Q}{2\pi T}\left[\infty - \ln r \int_0^\infty A(\alpha) J_o(\alpha r)\, d\alpha\right]$$

$$\int_0^\infty A(\alpha) J_o(\alpha r)\, d\alpha = \infty - \ln r$$

Given the relation

$$\int_0^\infty e^{-\alpha b} J_o(\alpha r)\, d\alpha = \frac{1}{\sqrt{b^2 + r^2}}$$

Integrate both sides from zero to ∞, w.r. to b.

$$\int_0^\infty db \int_0^\infty e^{-\alpha b} J_o(\alpha r)\, d\alpha = \int_0^\infty \frac{1}{\sqrt{b^2 + r^2}}\, db$$

$$\int_0^\infty J_o(\alpha r)\, d\alpha \int_0^\infty e^{-\alpha b}\, db = \ln(b + \sqrt{b^2 + r^2}) \Big]_0^\infty$$

$$\int_0^\infty \frac{J_{o(\alpha r)}}{\alpha}\, d\alpha = \infty - \ln r$$

$$\therefore \int_0^\infty A(\alpha) J_o(\alpha r)\, d\alpha = \int_0^\infty \frac{J_{o(\alpha r)}}{\alpha}\, d\alpha$$

$$A(\alpha) = \frac{1}{\alpha}$$

$$s = \frac{Q}{2\pi T}\left[\int_0^{\infty} \frac{J_{o(\alpha r)}}{\alpha}\, d\alpha - \int_0^{\infty} \frac{1}{\alpha} J_o(\alpha r) e^{-\nu\alpha^2 t}\, d\alpha\right]$$

$$= \frac{Q}{2\pi T}\int_0^{\infty} (1-e^{-\nu\alpha^2 t}) \frac{J_{o(\alpha r)}}{\alpha}\, d\alpha$$

Given the relation

$$\int_0^{\infty} \alpha e^{-y\alpha^2} J_o(\alpha r) d\alpha = \frac{e^{-\frac{r^2}{4y}}}{2y}$$

Integrate w.r to y from zero to νt

$$\int_0^{\nu t}\left[\int_0^{\infty} e^{-y\alpha^2} J_o(\alpha r) d\alpha\right] dy = \int_0^{\nu t} \frac{e^{-\frac{r^2}{4y}}}{y}\, dy$$

$$\int_0^{\infty} \alpha\, J_o(\alpha r) \frac{\left[-e^{-y\alpha^2}\right]_0^{\nu t}}{\alpha^2}\, d\alpha = \frac{1}{2}\int_0^{\nu t} \frac{e^{-\frac{r^2}{4y}}}{y}\, dy$$

$$\int_0^{\infty} \frac{1-e^{-\nu\alpha^2 t}}{\alpha} J_o(\alpha r) d\alpha = \frac{1}{2}\int_0^{\nu t} \frac{e^{-\frac{r^2}{4y}}}{y}\, dy$$

$$\therefore s = \frac{Q}{4\pi T}\int_0^{\nu t} \frac{e^{-\frac{r^2}{4y}}}{y}\, dy$$

Let $\frac{r^2}{4y} = \beta$, then $\frac{dy}{y} = -\frac{d\beta}{\beta}$

Limits @ $y = \nu t$ $\qquad \beta = \frac{r^2}{4\nu t}$

@ $y = 0$ $\qquad \beta = \infty$

$$s = \frac{Q}{4\pi T}\int_{\infty}^{\frac{r^2}{4\nu t}} e^{-\beta}\left(-\frac{d\beta}{\beta}\right)$$

or

$$s = \frac{Q}{4\pi T}\int_{\frac{r^2}{4\nu t}}^{\infty} \frac{e^{-\beta}}{\beta}\, d\beta$$

$$= \frac{Q}{4\pi T}\left[-Ei\left(-\frac{r^2}{4\nu t}\right)\right]$$

Or

$$s = \frac{Q}{4\pi T} W\left(\frac{r^2}{4\nu t}\right)$$

$$= \frac{Q}{4\pi T} W(u) \qquad \text{Theis Equation}$$

where $\qquad u = \frac{r^2}{4\pi t}$

$W(u)$ = Well function = $-Ei(-u)$

$-Ei(-u)$ = The negative exponential integral of $(-u) = \int_u^{\infty} \frac{e^{-\beta}}{\beta} \, d\beta$

For $u < 0.02$

$$W(u) = -\gamma - \ln u = \ln \frac{2.25\,\nu t}{r^2}$$

and

$$s = \frac{Q}{4\pi T} \ln \frac{2.25\,\nu t}{r^2}$$

SOLUTION OF PROBLEMS BY THE METHOD OF LAPLACE TRANSFORMATION

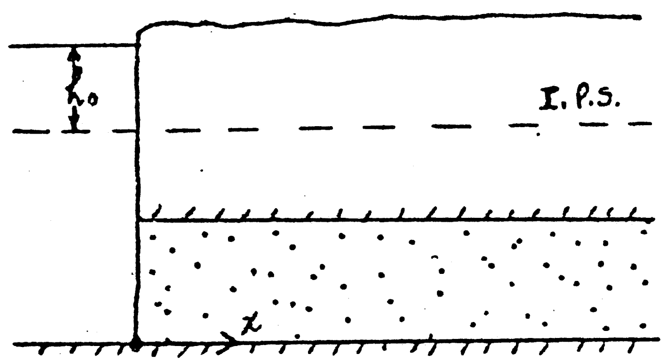
Flow from a line source in a confined aquifer

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu} \frac{\partial h}{\partial t} \qquad (1)$$

$$h(x,0) = 0 \qquad (2)$$

$$h(0,t) = h_o \qquad (3)$$

$$h(\infty,t) = 0 \qquad (4)$$



Transforming and using (2)

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{p}{\nu} \bar{h} \qquad (5)$$

$$\bar{h}(0,p) = \frac{h_o}{p} \qquad (6)$$

$$\bar{h}(\infty,p) = 0 \qquad (7)$$

From (5) $\bar{h} = c_1 e^{-\sqrt{\frac{p}{\nu}}x} + c_2 e^{\sqrt{\frac{p}{\nu}}x}$

From (7) $c_2 = 0$

From (6) $c_1 = \frac{h_o}{p}$

$$\bar{h} = \frac{h_o}{p} e^{-\sqrt{\frac{p}{\nu}}x}$$

$$L^{-1}\left\{\frac{e^{-\sqrt{\frac{p}{\nu}}x}}{p}\right\} = \text{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

$$h = h_o \,\text{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

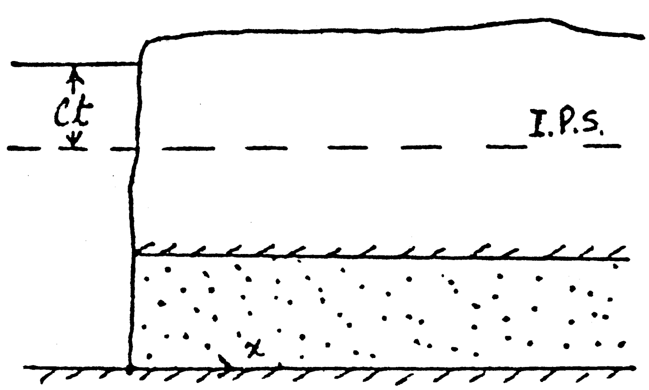
Flow from a nonsteady line source

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(x,0) = 0 \qquad (2)$$

$$h(0,t) = ct \qquad (3)$$

$$h(\infty,t) = 0 \qquad (4)$$



The transformed problem will be:

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{p}{\nu} \bar{h} \qquad (5)$$

$$\bar{h}(0,p) = \frac{c}{p^2} \qquad (6)$$

$$\bar{h}(\infty,p) = 0 \qquad (7)$$

From (5) $\bar{h} = c_1 \, e^{-x\sqrt{\frac{p}{\nu}}} + c_2 \, e^{x\sqrt{\frac{p}{\nu}}}$

From (7) $c_2 = 0$

From (6) $c_1 = \frac{c}{p^2}$

$$\bar{h} = \frac{c}{p^2} e^{-x\sqrt{\frac{p}{\nu}}}$$

$$L^{-1} \left\{ \frac{1}{p} \right\} = 1$$

$$L^{-1} \left\{ \frac{e^{-x\sqrt{\frac{p}{\nu}}}}{p} \right\} = \text{erfc} \left(\frac{x}{\sqrt{4\nu t}} \right)$$

By the convolution theorem

$$h(x,t) = c \int_0^t \text{erfc} \left(\frac{x}{\sqrt{4\nu\tau}} \right) d\tau$$

$$h(x,t) = ct\left\{(1+2\sqrt{u})\ \text{erfc}(\sqrt{u}) - \frac{2}{\sqrt{\pi}}\sqrt{u}\ e^{-u}\right\}$$

$$= 4\ ct\ i^2\ \text{erfc}\ (\sqrt{u})$$

where

$$u = \frac{x^2}{4\nu t}$$

$$i^2\text{erfc}(\beta) = \text{The second repeated integral of erfc}(\beta)$$

In general

$$i^n\ \text{erfc}(\beta) = \int_{\beta}^{\infty} i^{n-1}\ \text{erfc}(\beta)\ d\beta$$

The solution could have been obtained without the convolution from the following important inverse transform, not usually listed in tables of inverse Laplace transforms

$$L^{-1}\left\{\frac{e^{-K\sqrt{p}}}{p^{1+\frac{n}{2}}}\right\} = (4t)^{\frac{n}{2}}\ i^n\ \text{erfc}\left(\frac{K}{2\sqrt{t}}\right)$$

<u>A general formula for flow from a nonsteady line source</u>

In the two previous problems the variation of head in the line source was given as h_o and ct respectively. A more general case will be when the head in the line source starts varying as any function of time f(t).

The boundary value problem will be:

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(x,0) = 0 \qquad (2)$$

$$h(0,t) = f(t) \qquad (3)$$

$$h(\infty,t) = 0 \qquad (4)$$

The transformed problem:

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{p}{\nu}\,\bar{h} \qquad (5)$$

$$\bar{h}(0,p) = \bar{f}(p) \qquad (6)$$

$$\bar{h}(\infty,p) = 0 \qquad (7)$$

From similarity to the two previous problems:

$$\bar{h} = \bar{f}(p)\; e^{-x\sqrt{\frac{p}{\nu}}}$$

$$L^{-1}\left\{\bar{f}(p)\right\} = f(t)$$

$$L^{-1}\left\{e^{-x\sqrt{\frac{p}{\nu}}}\right\} = \frac{x}{\sqrt{4\nu\pi t^3}}\; e^{-\frac{x^2}{4\nu t}}$$

By the convolution theorem:

$$h(x,t) = \frac{x}{2\sqrt{\nu\pi}} \int_0^t f(t-\tau)\, \frac{e^{-\frac{x^2}{4\nu\tau}}}{\tau^{3/2}}\, d\tau$$

Let $\frac{x^2}{4\nu\tau} = \beta^2$, then $\frac{d\tau}{\tau} = -2\,\frac{d\beta}{\beta}$

$\tau = \dfrac{\quad}{4\nu\beta^2}$

Limits $\quad @\ \tau = t \qquad \beta = \frac{x}{\sqrt{4\nu t}}$

$\qquad\quad @\ \tau = 0 \qquad \beta = \infty$

$$h(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} f\left(t - \frac{x^2}{4\nu\beta^2}\right) e^{-\beta^2}\, d\beta$$

As an example, when $f(t) = h_o$

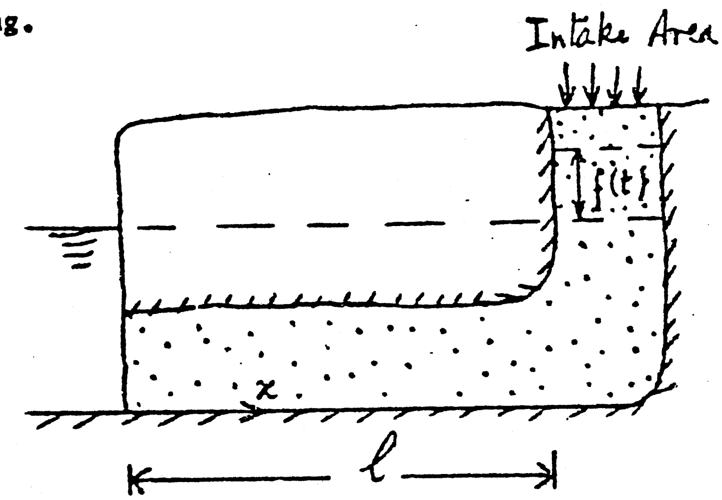
$$h(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} h_o\, e^{-\beta^2}\, d\beta$$

$$= h_o\, \mathrm{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)$$

$$\frac{X}{4\frac{T}{S}}$$

Flow between two line scurces

Due to recharge the water level in the intake area starts rising.



$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(x\ ,0) = 0 \qquad (2)$$

$$h(0,t) = 0 \qquad (3)$$

$$h(\ell,t) = f(t) \qquad (4)$$

Transforming:

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{p}{\nu}\bar{h} \qquad (5)$$

$$\bar{h}(0,p) = 0 \qquad (6)$$

$$\bar{h}(\ell,p) = \bar{f}(p) \qquad (7)$$

Due to the limited length we prefer hyperbolic functions, instead of exponentials

From (5) $\bar{h} = c_1 \sinh x\sqrt{\frac{p}{\nu}} + c_2 \cosh x\sqrt{\frac{p}{\nu}}$

From (6) $c_2 = 0$

From (7) $c_1 = \dfrac{\bar{f}(p)}{\sinh \ell\sqrt{\frac{p}{\nu}}}$

$$\bar{h} = \bar{f}(p) \; \frac{\sinh x\sqrt{\frac{p}{\nu}}}{\sinh \ell\sqrt{\frac{p}{\nu}}}$$

$$= \bar{f}(p) \; \frac{e^{x\sqrt{\frac{p}{\nu}}} - e^{-x\sqrt{\frac{p}{\nu}}}}{e^{\ell\sqrt{\frac{p}{\nu}}} - e^{-\ell\sqrt{\frac{p}{\nu}}}}$$

$$= \bar{f}(p) \; e^{-\ell\sqrt{\frac{p}{\nu}}} \; \frac{\left(e^{x\sqrt{\frac{p}{\nu}}} - e^{-x\sqrt{\frac{p}{\nu}}}\right)}{1-e^{-2\ell\sqrt{\frac{p}{\nu}}}}$$

But $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$

$$\therefore \; \bar{h} = \bar{f}(p)\left[e^{-(\ell-x)\sqrt{\frac{p}{\nu}}} - e^{-(\ell+x)\sqrt{\frac{p}{\nu}}}\right]\sum_{n=0}^{\infty} e^{-2n\ell\sqrt{\frac{p}{\nu}}}$$

$$= \bar{f}(p) \sum_{n=0}^{\infty} e^{-\sqrt{\frac{p}{\nu}}\,(\ell-x+2n\ell)} - e^{-\sqrt{\frac{p}{\nu}}\,(\ell+x+2n\ell)}$$

If $f(t) = h_o$, $\bar{f}(p) = \frac{h_o}{p}$

$$\bar{h} = h_o \sum_{n=0}^{\infty} \frac{e^{-[(2n+1)\ell-x]\sqrt{\frac{p}{\nu}}}}{p} - \frac{e^{-[(2n+1)\ell+x]\sqrt{\frac{p}{\nu}}}}{p}$$

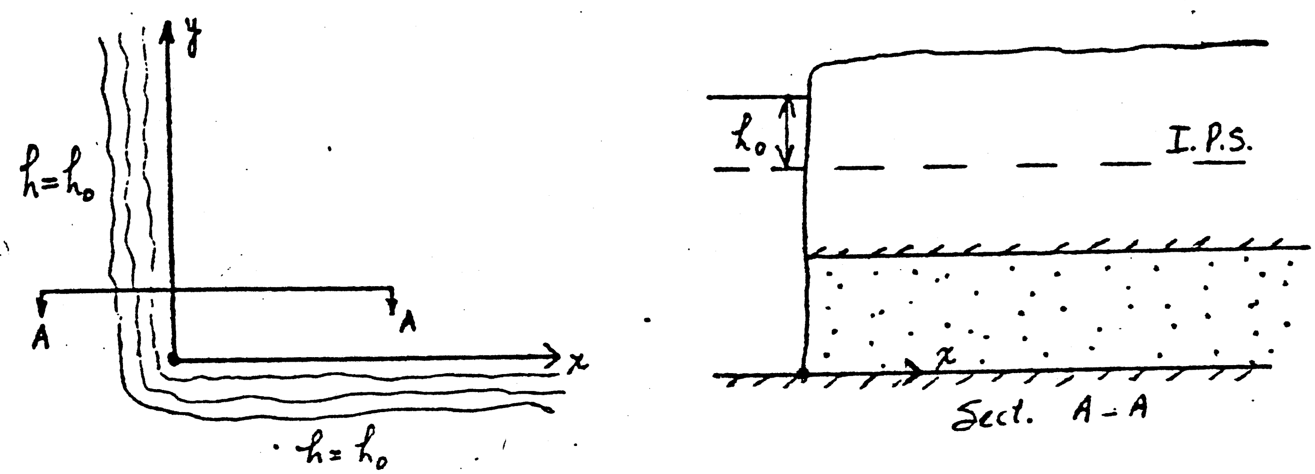
and

$$h = h_o \sum_{n=0}^{\infty} \operatorname{erfc}\left(\frac{(2n+1)\ell-x}{\sqrt{4\nu t}}\right) - \operatorname{erfc}\left(\frac{(2n+1)\ell+x}{\sqrt{4\nu t}}\right)$$

Let $n = m-1$

$$h = h_o \sum_{n=1}^{\infty} \operatorname{erfc}\left(\frac{(2m-1)\ell-x}{\sqrt{4\nu t}}\right) - \operatorname{erfc}\left(\frac{(2m-1)\ell+x}{\sqrt{4\nu t}}\right)$$

Flow between two line sources intersecting each other at right angles.-



$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(0,y,t) = h(x,0,t) = h_o \qquad (2)$$

$$h(x,y,0) = 0 \qquad (3)$$

$$h(\infty,\infty,t) = 0 \qquad (4)$$

Assuming a product solution of the form: $h(x,y,t) = X(x,t)Y(y,t)$ and from the previous solutions of unidirectional flow, the solution is built up.

$$h = h_o\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)\operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

A check shows that all the boundary conditions are satisfied.

To obtain a general solution for a variation of condition (2) as

$$h(0,y,t) = h(x,0,t) = f(t)$$

we proceed as follows

For $f(t) = h_o = c$

$$h = c\left[1 - \text{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

$$\bar{h}(x,y,p) = c\ L\left\{1 - \text{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right\}$$

$$= c\ \frac{V(p)}{p}$$

Since $L\left\{c\right\} = \frac{c}{p}$, if $f(t) \neq c$, then $\bar{h} = \bar{f}(p)\ V(p)$

or $$\bar{h} = \bar{f}(p)\ p\left(\frac{V(p)}{p}\right)$$

Remembering that multiplying by p corresponds to differentiating the untransformed function with respect to t

$$L^{-1}\left\{p\left(\frac{V(p)}{p}\right)\right\} = \frac{\partial}{\partial t}\left[1 - \text{erf}(\ )\ \text{erf}(\ )\right]$$

and by the convolution

$$h(x,y,t) = \int_0^t f(\tau)\, \frac{\partial}{\partial t}\, [1-\text{erf}(\ )\ \text{erf}(\ )]_{t=t-\tau}\, d\tau$$

or

$$= \int_0^t f(t-\tau)\, \frac{\partial}{\partial t}\, [1-\text{erf}(\ )\ \text{erf}(\ )]_{t=\tau}\, d\tau$$

This formula is known as the Duhamel formula in mathematical literature.

As an application let's take $f(t) = ct$

$$h(x,y,t) = c \int_0^t \tau \frac{\partial}{\partial t} \left[1-\text{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]_{t=t-\tau} d\tau$$

$$= c \int_0^t \tau \left\{ -\text{erf}\left(\frac{x}{\sqrt{4\nu(t-\tau)}}\right) \cdot \frac{2}{\sqrt{\pi}}\ e^{-\frac{y^2}{4\nu(t-\tau)}} \cdot \left(-\frac{1}{2} \frac{y}{\sqrt{4\nu(t-\tau)^3}}\right)\right.$$

$$\left. - \text{erf}\left(\frac{y}{\sqrt{4\nu(t-\tau)}}\right) \cdot \frac{2}{\sqrt{\pi}}\ e^{-\frac{x^2}{4\nu(t-\tau)}} \cdot \left(-\frac{1}{2} \frac{x}{\sqrt{4\nu(t-\tau)^3}}\right)\right\} d\tau$$

$$= \frac{c}{2\sqrt{\pi\nu}} \int_0^t \left(y\tau\ \frac{e^{-\frac{y^2}{4\nu(t-\tau)}}}{\sqrt{(t-\tau)^3}}\ \text{erf}\left(\frac{x}{\sqrt{4\nu(t-\tau)}}\right)\right.$$

$$\left. +\ x\tau\ \frac{e^{-\frac{x^2}{4\nu(t-\tau)}}}{\sqrt{(t-\tau)^3}}\ \text{erf}\left(\frac{y}{\sqrt{4\nu(t-\tau)}}\right)\right) d\tau$$

Let

$$\frac{x^2}{4\nu(t-\tau)} = \beta_1^2$$

and
$$\frac{y^2}{4\nu(t-\tau)} = \beta_2^2$$

$$h(x,y,t) = \frac{cy}{\sqrt{4\pi\nu}} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} \left[t - \frac{x^2}{4\nu\beta_1^2}\right] e^{-\left(\frac{y}{x}\right)^2 \beta_1^2} \operatorname{erf}(\beta_1) \frac{4\sqrt{\nu}}{x} d\beta_1$$

$$+ \frac{cx}{\sqrt{4\pi\nu}} \int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} \left[t - \frac{y^2}{4\nu\beta_1^2}\right] e^{-\left(\frac{x}{y}\right)^2 \beta_2^2} \operatorname{erf}(\beta_2) \frac{4\sqrt{\nu}}{y} d\beta_2$$

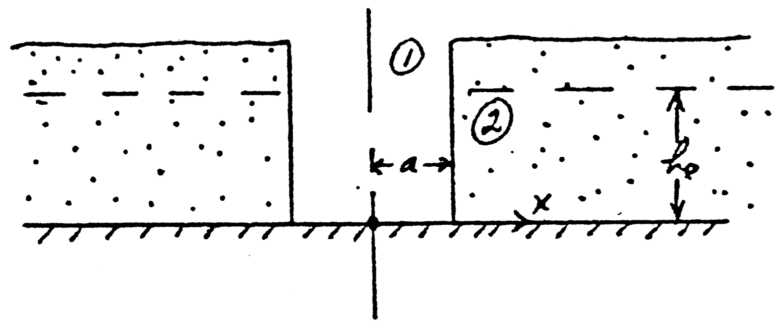
$$h(x,y,t) = \frac{2cy}{\sqrt{\pi}x} \int_{\frac{x}{\sqrt{4\nu t}}}^{\infty} \left(t - \frac{x^2}{4\nu\beta_1^2}\right) e^{-\left(\frac{y}{x}\right)^2 \beta_1^2} \operatorname{erf}(\beta_1)\, d\beta_1$$

$$+ \frac{2cx}{\sqrt{\pi}y} \int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} \left(t - \frac{y^2}{4\nu\beta_2^2}\right) e^{-\left(\frac{x}{y}\right)^2 \beta_2^2} \operatorname{erf}(\beta_2)\, d\beta_2$$

In the same manner the problem can be solved for different f(t).

Seepage into a ditch

We will first consider the case where the ditch is not excavated yet, but assuming that by one way or other the water is drained from region (1)



In region (1)

$$\frac{\partial^2 h_1}{\partial x^2} = \frac{1}{\nu}\frac{\partial h_1}{\partial t} \qquad (1)$$

$\nu = \frac{K}{S_s}$

$$h_1(x,0) = 0 \qquad (2)$$

$$\frac{\partial h_1}{\partial x}(0,t) = 0 \qquad (3)$$

In region (2)

$$\frac{\partial^2 h_2}{\partial x^2} = \frac{1}{\nu}\frac{\partial h_2}{\partial t} \qquad (4)$$

$$h_2(x,0) = h_o \qquad (5)$$

$$h_2(\infty,t) = h_o \qquad (6)$$

Common conditions

$$h_1(a,t) = h_2(a,t) \qquad (7)$$

$$\frac{\partial h_1}{\partial x}(a,t) = \frac{\partial h_2}{\partial x}(a,t) \qquad (8)$$

The transformed problem will be

In (1)

$$\frac{\partial^2 \bar{h}_1}{\partial x^2} = \frac{p}{\nu}\ \bar{h}_1 \qquad (9)$$

$$\frac{\partial \bar{h}_1}{\partial x}(0,p) = 0 \qquad (10)$$

In (2)

$$\frac{\partial^2 \bar{h}_2}{\partial x^2} = \frac{p}{\nu}\bar{h}_2 - \frac{h_o}{\nu} \qquad (11)$$

$$\bar{h}_2\ (\infty,p) = \frac{h_o}{p} \qquad (12)$$

Common conditions

$$\bar{h}_1(a,p) = \bar{h}_2(a,p) \qquad (13)$$

$$\frac{\partial \bar{h}_1}{\partial x}(a,p) = \frac{\partial \bar{h}_2}{\partial x}\ (a,p) \qquad (14)$$

Solving (9) and choosing a hyperbolic function because it applies to a limited area, and satisfying (10)

$$\bar{h}_1 = c_1 \cosh x\sqrt{\frac{p}{\nu}}$$

From (11) & (12)

$$\bar{h}_2 = c_2\, e^{-x\sqrt{\frac{p}{\nu}}} + \frac{h_o}{p}$$

From (13)

$$c_1 \cosh a\sqrt{\frac{p}{\nu}} - c_2\, e^{-a\sqrt{\frac{p}{\nu}}} = \frac{h_o}{p}$$

From (14)

$$c_1 \sinh a\sqrt{\frac{p}{\nu}} + c_2\, e^{-a\sqrt{\frac{p}{\nu}}} = 0$$

Therefore

$$c_1 = \frac{h_o}{p}\, e^{-a\sqrt{\frac{p}{\nu}}}$$

$$c_2 = -\frac{h_o}{p} \sinh a\sqrt{\frac{p}{\nu}}$$

and

$$\bar{h}_1 = \frac{h_o}{p}\, e^{-a\sqrt{\frac{p}{\nu}}} \cosh x\sqrt{\frac{p}{\nu}}$$

$$= \frac{h_o}{2p}\left(e^{-(a-x)\sqrt{\frac{p}{\nu}}} + e^{-(a+x)\sqrt{\frac{p}{\nu}}}\right)$$

$$\bar{h}_2 = \frac{h_o}{p}\left[1 - \frac{1}{2}\left(e^{-(x-a)\sqrt{\frac{p}{\nu}}} - e^{-(x+a)\sqrt{\frac{p}{\nu}}}\right)\right]$$

Taking the inverse transforms

$$h_1 = \frac{h_o}{2}\left[\text{erfc}\left(\frac{a-x}{\sqrt{4\nu t}}\right) + \text{erfc}\left(\frac{a+x}{\sqrt{4\nu t}}\right)\right]$$

$$h_2 = \frac{h_o}{2}\left[2 - \text{erfc}\left(\frac{x-a}{\sqrt{4\nu t}}\right) + \text{erfc}\left(\frac{x+a}{\sqrt{4\nu t}}\right)\right]$$

But

$$\text{erfc}(-x) = 1 - \text{erf}(-x) = 1 + \text{erf}(x) = 1 + (1-\text{erfc}(x) = 2 - \text{erfc}(x)$$

and

$$h_2 = \frac{h_o}{2}\left[\text{erfc}\left(\frac{a-x}{\sqrt{4\nu t}}\right) + \text{erfc}\left(\frac{a+x}{\sqrt{4\nu t}}\right)\right]$$

We see that h_1 and h_2 are given by the same expression. Although h_1 was solved for region (1) and h_2 for region (2), the same expression will give the head in the whole region. The change in sign of $(a-x)$ for $x > a$ will take care of the difference in the boundary conditions. Actually the expression for h_1 changes to that for h_2 if $x > a$ since erfc $(-x) = 2 -$ erfc(x).

To calculate the rate of seepage:

$$h = \frac{h_o}{2}\left[\text{erfc}\left(\frac{a-x}{\sqrt{4\nu t}}\right) - \text{erfc}\left(\frac{a+x}{\sqrt{4\nu t}}\right)\right]$$

$$q = T \left.\frac{\partial h}{\partial x}\right|_{x=a}$$

$$q = \frac{Th_o}{2}\left[\frac{2}{2\sqrt{\nu\pi t}}\left(e^{-\frac{(a-x)^2}{4\nu t}} - e^{-\frac{(a+x)^2}{4\nu t}}\right)^2\right]_{x=a}$$

$$q = \frac{Th_o}{2\sqrt{\nu\pi t}}\left(1 - e^{-\frac{a^2}{\nu t}}\right)$$

Now we will solve the problem for the case when the ditch is excavated and empty and the water level starts to rise in the initially dry ditch as an unknown function of time:

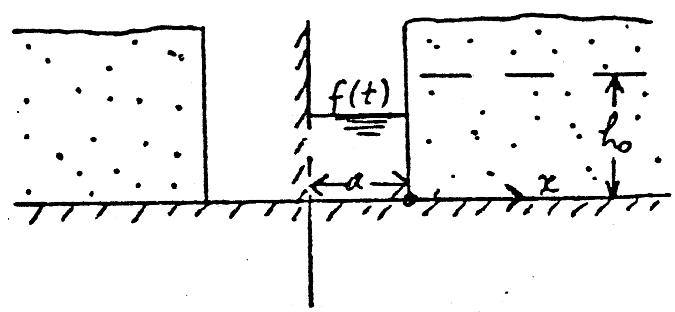
$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \qquad (1)$$

$$h(x,0) = h_o \qquad (2)$$

$$h(\infty,t) = h_o \qquad (3)$$

$$h(0,t) = f(t) \qquad (4)$$

$$f(0) = 0 \qquad (5)$$



Transforming

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{p}{\nu}\bar{h} - \frac{h_o}{\nu} \qquad (6)$$

$$\bar{h}(\infty,p) = \frac{h_o}{p} \qquad (7)$$

$$\bar{h}(0,p) = \bar{f}(p) \qquad (8)$$

$$\bar{f}(0) = 0 \qquad (9)$$

From (6) and (7)

$$\bar{h} = c_1 \, e^{-x\sqrt{\frac{p}{\nu}}} + \frac{h_o}{p}$$

From (8)

$$\bar{f}(p) - \frac{h_o}{p} = c_1$$

$$\bar{h} = \left(\bar{f}(p) - \frac{h_o}{p}\right) e^{-x\sqrt{\frac{p}{\nu}}} + \frac{h_o}{p}$$

The rate of increase of the water level in the ditch is related to the discharge

$$a \frac{\partial f(t)}{\partial t} = T \left.\frac{\partial h}{\partial x}\right|_{x=0}$$

Transforming

$$a \, p \, \bar{f}(p) = T \left.\frac{\partial \bar{h}}{\partial x}\right|_{x=0}$$

$$= T\left(-\sqrt{\frac{p}{\nu}}\right)\left(\bar{f}(p) - \frac{h_o}{p}\right)$$

$$p \, \bar{f}(p) = -\delta\left(\bar{f}(p) - \frac{h_o}{p}\right)\sqrt{p}$$

where $\delta = \frac{T}{a\sqrt{\nu}}$

$$\bar{f}(p) = \frac{\delta h_o}{p(\delta+\sqrt{p})}$$

$$L^{-1}\left\{\frac{\delta e^{-K\sqrt{p}}}{p(\delta+\sqrt{p})}\right\} = \left(-e^{-\delta K}\, e^{\delta^2 t}\, \text{erfc}(\delta\sqrt{t} + \frac{K}{\sqrt{4t}}\right)$$

$$+ \text{erfc}\left(\frac{K}{\sqrt{4t}}\right)\Bigg) \quad \text{for } K \geq 0$$

$$f(t) = h_o\,(1-e^{\delta^2 t}\,\text{erfc}(\delta\sqrt{t})$$

$$\bar{h} = \frac{\delta h_o}{p(\delta+\sqrt{p})}\, e^{-x\sqrt{\frac{p}{\nu}}} + h_o\left(\frac{1}{p} - \frac{e^{-x\sqrt{\frac{p}{\nu}}}}{p}\right)$$

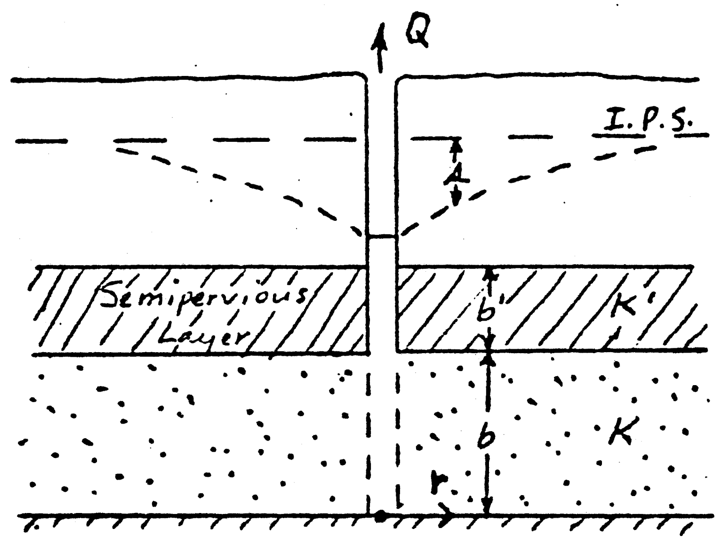
$$h = h_o\, \text{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right) - h_o\, e^{-\frac{\delta x}{\sqrt{\nu}}}\, e^{\delta^2 t}\, \text{erfc}\left(\delta\sqrt{t} + \frac{x}{\sqrt{4\nu t}}\right)$$

$$+ h_o\left(1 - \text{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)\right)$$

$$h = h_o\left[1 - e^{-\frac{\delta x}{\sqrt{\nu}}}\, e^{\delta^2 t}\, \text{erfc}\left(\delta\sqrt{t} + \frac{x}{\sqrt{4\nu t}}\right)\right]$$

Flow to a well in a confined aquifer

1. Leaky Aquifer



$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (1)$$

$$s(r,0) = 0 \qquad (2)$$

$$s(\infty,t) = 0 \qquad (3)$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \qquad (4)$$

Transforming

$$\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r} - \frac{\bar{s}}{B^2} = \frac{p}{\nu}\bar{s} \qquad (5)$$

$$\bar{s}(\infty,p) = 0 \qquad (6)$$

$$\lim_{r \to 0} r\frac{\partial \bar{s}}{\partial r} = -\frac{Q}{2\pi T p} \qquad (7)$$

From (5)

$$\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r} - \left(\frac{1}{B^2} + \frac{p}{\nu}\right)\bar{s} = 0 \quad \text{(Mod. Bessel Eq.)}$$

$$\bar{s} = c_1 I_o\left(r\sqrt{\frac{1}{B^2} + \frac{p}{\nu}}\right) + c_2 K_o\left(r\sqrt{\frac{1}{B^2} + \frac{p}{\nu}}\right)$$

From (6) $c_1 = 0$

From (7) $c_2 = \frac{Q}{2\pi T p}$

$$\bar{s} = \frac{Q}{2\pi T p} K_o\left(r\sqrt{\frac{1}{B^2} + \frac{p}{\nu}}\right)$$

$$s = \frac{Q}{2\pi T} L^{-1}\left\{\frac{1}{p} \cdot Ko\left(r\sqrt{\frac{1}{B^2} + \frac{p}{\nu}}\right)\right\}$$

$$L^{-1}\left\{\frac{1}{p}\right\} = 1$$

$$L^{-1}\left\{K_o\left(r\sqrt{\frac{1}{B^2} + \frac{p}{\nu}}\right)\right\} = L^{-1}\left\{K_o\left(\sqrt{\frac{\nu}{B^2} + p}\ \frac{r}{\sqrt{\nu}}\right)\right\}$$

$$= e^{-\frac{\nu}{B^2}t}\ \frac{1}{2t}\ e^{-\frac{r^2}{4\nu t}}$$

By convolution

$$s = \frac{Q}{4\pi T}\int_0^t \frac{e^{-\left(\frac{\nu}{B^2}\tau + \frac{r^2}{4\nu\tau}\right)}}{\tau}\,d\tau$$

The substitution $y = \frac{r^2}{4\nu\tau}$ leads to:

$$s = \frac{Q}{4\pi T} \int_{\frac{r^2}{4\nu t}}^{\infty} \frac{e}{y}^{-y-\frac{r^2/B^2}{4y}} \, dy$$

$$s = \frac{Q}{4\pi T} W(u,\frac{r}{B}) \qquad \text{Hantush formula.}$$

$$u = \frac{r^2}{4\pi t} \; ; \; W(u,\frac{r}{B}) = \int_u^{\infty} \frac{e}{y}^{-y-\frac{r^2/B^2}{4y}} \, dy$$

$W(u,\frac{r}{B})$ is known as the « well function for leaky aquifers », and is tabulated.

1. Nonleaky Aquifers:

The solution can be obtained in a similar way, but also be letting $\frac{1}{B} \to 0$ in the solution for leaky aquifers.

$$s = \frac{Q}{4\pi T} \int_u^{\infty} \frac{e}{y}^{-y-\frac{r^2/B^2}{4y}} \, dy$$

as $\frac{1}{B} \to 0$

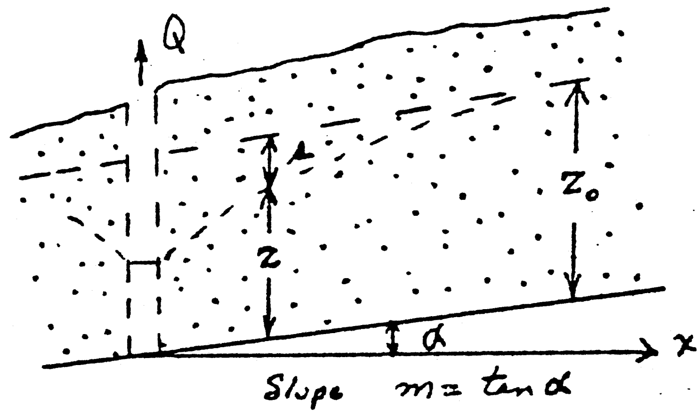
$$s = \frac{Q}{4\pi T} \int_u^{\infty} \frac{e^{-y}}{y} \, dy$$

$$s = \frac{Q}{4\pi T} W(u)$$

Flow to a well in an inclined unconfined aquifer

$$\frac{\partial^2 z^2}{\partial x^2} + \frac{m}{\bar{z}}\frac{\partial z^2}{\partial x} + \frac{\partial^2 z^2}{\partial y^2} = \frac{S_y}{K\bar{z}}\frac{\partial z^2}{\partial t} \qquad (1)$$

$$z(x,y,0) = z_o \qquad (2)$$



$$z(\infty,\infty,t) = z_o \qquad (3)$$

$$\lim_{r \to 0} 2rz\frac{\partial z}{\partial r} = \frac{Q}{\pi K}$$

$$\text{or} \lim_{r \to 0} r\frac{\partial z^2}{\partial r} = \frac{Q}{\pi K} \qquad (4)$$

The substitution $z^2 = \omega e^{-\frac{r}{\beta}\cos\theta}$ in (1) leads to:

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} - \frac{\omega}{\beta^2} = \frac{1}{\nu}\frac{\partial \omega}{\partial t}$$

where, $\beta = \frac{2\bar{z}}{m}$; $r = x^2+y^2$; $\theta = \tan^{-1}\frac{y}{x}$; $\frac{1}{\nu} = \frac{S_y}{K\bar{z}}$;

in cylindrical coordinates:

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} - \frac{\omega}{\beta^2} = \frac{1}{\nu}\frac{\partial \omega}{\partial t}$$

(2) becomes $\omega(r,\theta,0) = z_o^2 e^{\frac{r}{\beta}\cos\theta}$

By Laplace transformation

$$\frac{\partial^2 \bar{\omega}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\omega}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\omega}}{\partial \theta^2} - \frac{\bar{\omega}}{\beta} = \frac{p}{\nu}\bar{\omega} - \frac{z_o^2}{\nu} e^{\frac{r}{\beta}\cos\theta}$$

A solution satisfying this equation and cond. (3) is:

$$\bar{\omega} = c_1 K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{\beta^2}}\right) + \frac{z_o^2}{p} e^{\frac{r}{\beta}\cos\theta}$$

or

$$\bar{z}^2 = c_1 e^{-\frac{r}{\beta}\cos\theta} K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{\beta^2}}\right) + \frac{z_o^2}{p}$$

Cond. (4) transforms to:

$$\lim_{r \to 0} r \frac{\partial \bar{z}^2}{\partial r} = \frac{Q}{\pi K p}$$

Therefore $$c_1 = -\frac{Q}{\pi K p}$$

$$z^2 = \frac{z_o^2}{p} - \frac{Q}{\pi K p} e^{-\frac{r}{\beta}\cos\theta} K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{\beta^2}}\right)$$

$$z_o^2 - z^2 = \frac{Q}{\pi K} e^{-\frac{r}{\beta}\cos\theta} L^{-1}\left\{\frac{1}{p} K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{\beta^2}}\right)\right\}$$

From similarity to the $L^{-1}\{\}$ encountered in the problem of flow to a well in a leaky aquifer (see page 109)

$$z_o^2 - z^2 = \frac{Q}{2\pi K} e^{-\frac{r}{\beta}\cos\theta} W(u, \frac{r}{\beta})$$

In terms of drawdown

$$s - \frac{s^2}{2z_o} = \frac{Q}{4\pi K z_o} e^{-\frac{r}{\beta}\cos\theta} W(u, \frac{r}{\beta})$$

For small $\frac{s}{z_o}$, i.e. when the drawdown is small compared to the total thickness

$$s = \frac{Q}{4\pi K z_o} e^{-\frac{r}{\beta}\cos\theta} W(u, \frac{r}{\beta})$$

Tables of the previous by mentioned $W(u, \frac{r}{B})$ can be used to find

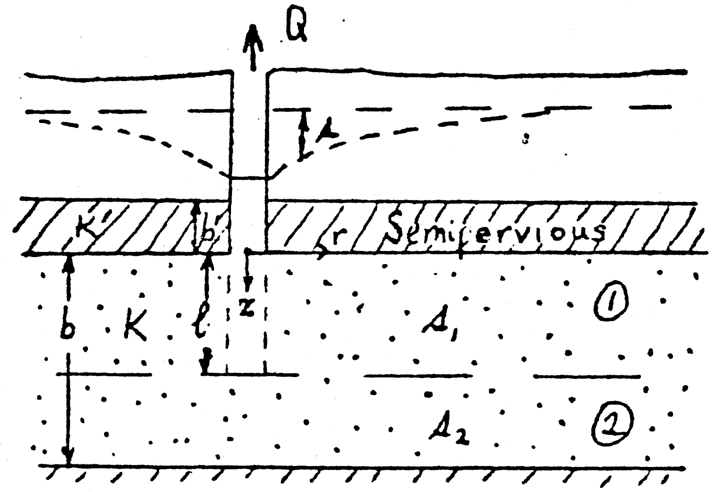
$W(u, \frac{r}{\beta})$, where $u = \frac{r^2}{4\nu t}$; $\beta = \frac{2\bar{z}}{m}$

<u>Flow to a partially penetrating well in a confined aquifer</u>

<u>Leaky Aquifer</u>

We separate the aquifer into two region ① & ②

The boundary value problem is given by



$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t} \quad (1)$$

$$s_2(r,z,0) = s_1(r,z,0) = 0 \quad (2)$$

$$s_2(\infty,z,t) = s_1(\infty,z,t) = 0 \qquad (3)$$

$$\frac{\partial s_2}{\partial z}(r,b,t) = \frac{\partial s_1}{\partial z}(r,0,t) = 0 \qquad (4)$$

$$\frac{\partial s_2}{\partial r}(0,z,t) = 0 \qquad (5)$$

$$\lim_{r \to 0} r\frac{\partial s_1}{\partial r} = -\frac{Q}{2\pi K\ell} \qquad (6)$$

$$s_1(r,\ell,t) = s_2(r,\ell,t) \qquad (7)$$

$$\frac{\partial s_1}{\partial z}(r,\ell,t) = \frac{\partial s_2}{\partial z}(r,\ell,t) \qquad (8)$$

By Laplace transformation

$$\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r} + \frac{\partial^2 \bar{s}}{\partial z^2} - \left(\frac{p}{\nu} + \frac{1}{B^2}\right)\bar{s} = 0$$

Solving by separation of variables and choosing from the particular solutions.

$$s_1 = c_1 K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{B^2}}\right) + \int_0^\infty A(\alpha)J_o(\alpha r)\cosh\left(z\sqrt{\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}}\right)d\alpha$$

$$s_2 = \int_0^\infty B(\alpha)J_o(\alpha r)\cosh\left((b-z)\sqrt{\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}}\right)d\alpha$$

From (6) $\qquad c_1 = \frac{Q}{2\pi K\ell p}$

From (7)

$$\frac{Q}{2\pi K\ell p} K_o(r\sqrt{\ }) + \int_0^\infty A(\alpha)\ J_o(\alpha r)\ \cosh(\ell\sqrt{\ })\ d\alpha =$$

$$= \int_0^\infty B(\alpha) J_o(\alpha r)\ \cosh\left((b-\ell)\sqrt{\ }\right) d\alpha$$

But $$K_o(rz) = \int_0^\infty \frac{\alpha J_o(\alpha r)}{\alpha^2+z^2}\ d\alpha$$

$$\int_0^\infty \left\{\frac{Q}{2\pi K\ell p} \cdot \frac{\alpha}{\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}} + A(\alpha)\ \cosh\ (\ell\sqrt{\ }) - B(\alpha)\cosh\left((b-\ell)\sqrt{\ }\right)\right\} J_o(\alpha r) d\alpha = 0$$

$$\therefore \frac{Q}{2\pi K\ell p}\ \frac{\alpha}{\alpha^2+ \frac{p}{\nu} + \frac{1}{B^2}} + A(\alpha)\ \cosh\ (\ell\sqrt{\ }) = B(\alpha)\ \cosh\ \left((b-\ell)\sqrt{\ }\right) \quad \text{(I)}$$

From (8)

$$\int_0^\infty A(\alpha) J_o(\alpha r)(\sqrt{\ })\ \sinh(\ell\sqrt{\ }) d\alpha = \int_0^\infty B(\alpha) J_o(\alpha r)(-\sqrt{\ })\sinh\ \left((b-\ell)\sqrt{\ }\right)\ d\alpha$$

$$\therefore \quad A(\alpha)\sinh(\ell\sqrt{\ }\ ) = -\ B(\alpha)\ \sinh\ \left((b-\ell)\sqrt{\ }\right) \quad \text{(II)}$$

After solving (I) & (II) for $A(\alpha)$ & $B(\alpha)$

$$\bar{s}_1 = \frac{Q}{2\pi K\ell} \left\{ \frac{K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{B^2}}\right)}{p} - \int_0^\infty c_1\ \frac{\alpha J_o(\alpha r)}{p\left(\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}\right)}\ d\alpha \right\}$$

$$\bar{s}_2 = \frac{Q}{2\pi K\ell} \int_0^\infty C_2 \; \frac{\alpha J_o(\alpha r)}{p\left(\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}\right)} \; d\alpha$$

where

$$c_1 = \frac{\sinh\left((b-\ell)\sqrt{\;}\right) \cosh(z\sqrt{\;})}{\sinh(b\sqrt{\;})}$$

$$c_2 = \frac{\sinh(\ell\sqrt{\;}) \cosh\left((b-z)\sqrt{\;}\right)}{\sinh(b\sqrt{\;})}$$

Expressing the hyperbolic functions in their exponential form, and using

$\frac{1}{1-z} = \sum\limits_{n=0}^{\infty} z^n$ for $|z| < 1$, we obtain

$$\bar{s}_1 = \frac{Q}{2\pi K\ell}\left[\frac{1}{p} K_o\left(r\sqrt{\frac{p}{\nu} + \frac{1}{B^2}}\right) - \int_0^\infty \frac{\alpha J_o(\alpha r)}{p\left(\alpha^2 + \frac{p}{\nu} + \frac{1}{B^2}\right)} \; d\alpha \right. \cdot$$

$$\cdot \sum_{n=0}^{\infty} \begin{array}{l} \exp(-[2nb+(\ell-z)]\sqrt{\;}) - \exp(-[2(n+1)b-(\ell+z)]\sqrt{\;}) \\ \exp(-[2nb\pm(\ell+z)]\sqrt{\;}) - \exp(-[2(n+1)b-(\ell-z)]\sqrt{\;}) \end{array}$$

Taking the inverse transform, by convolution and using the property

$$\int_0^\infty \alpha J_o(\alpha r) e^{-\mu\alpha^2} \, d\alpha = e^{-\frac{r^2}{4\mu}}$$

we obtain after some manipulation,

$$s_1 = \frac{Q}{8\pi K\ell} \int_u^\infty \frac{e^{-y-\frac{r^2}{4B^2 y}}}{y} \left[\operatorname{erf}\left(\frac{\ell - z}{r}\sqrt{y}\right) + \operatorname{erf}\left(\frac{\ell + z}{r}\sqrt{y}\right) \right] dy$$

$$+ \frac{Q}{8\pi K\ell} \sum_{n=1}^\infty \frac{e^{-y-\frac{r^2}{4B^2 y}}}{y} \cdot$$

$$\cdot \begin{bmatrix} \operatorname{erf}\left(\frac{2nb+(\ell-z)}{r}\sqrt{y}\right) - \operatorname{erf}\left(\frac{2nb-(\ell-z)}{r}\sqrt{y}\right) \\ \operatorname{erf}\left(\frac{2nb+(\ell+z)}{r}\sqrt{y}\right) - \operatorname{erf}\left(\frac{2nb-(\ell+z)}{r}\sqrt{y}\right) \end{bmatrix} dy$$

where

$$u = \frac{r^2}{4\nu t}$$

In the same manner the solution for s_2 can also be obtained.

Non-leaky Aquifer

For non-leaky aquifers $\frac{1}{B} = 0$, and the solution for s_1 is the same with the exception that the exponential terms change to e^{-y}.

Thick Aquifers:

For thick aquifers, $b > 100\ell$, or for aquifers of infinite thickness, $b = \infty$, the second part of the equation becomes zero.

$$s_1 = \frac{Q}{8\pi K\ell} \int_u^\infty \frac{e^{-y}}{y} \left[\operatorname{erf}\left(\frac{\ell - z}{r}\sqrt{y}\right) + \operatorname{erf}\left(\frac{\ell + z}{r}\sqrt{y}\right)\right] dy$$

$$= \frac{Q}{8\pi K\ell} \left[M\left(u, \frac{\ell - z}{r}\right) + M\left(u, \frac{\ell + z}{r}\right)\right]$$

where

$$M\left(u, \beta\right) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy$$

$M(u,\beta)$ is tabulated quite extensively in Professional Paper 102, NMIMT.